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TREATISE

ON

TRIGONOMETRY

BY

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PROFS. OLIVER, WAIT AND JONES

OF

CORNELL UNIVERSITY.

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PREFACE.

THIS book is designed as a drill-book for class use; its leading features are:

1. The general definition of the trigonometric functions in terms applicable to all angles, without regard to sign or magnitude.

2. The expression of the functions of all angles in terms of the functions of positive angles less than a right angle, by direct reference to the definitions.

3. The graphic representation of trigonometric functions.

4. The varied and repeated illustration of definitions and principles by carefully drawn figures.

5. The full discussion of the ambiguous and impossible cases of right and oblique triangles.

6. The differentiation of trigonometric functions, their development thereby into series, and the computation of the trigonometric canon by means of these series.

7. The exact statement of principles in the form of theorems and corollaries, and their rigorous demonstration.

8. Copious and varied exercises.

In the preparation of the book, free use has been made of the works of other authors, particularly those of Briot and Bouquet, De Morgan, Todhunter, Peirce, Wheeler, Green-

by merely reading demonstrations and solving illustrative numerical examples. To assimilate the principles of the subject, one must work them out afresh for himself, and discover their applications to new lines of thought. Counting upon this problem-solving quite as much as upon the reading of the text, the authors have sought to lay out such work as is most important and most instructive, with only so much of guidance as shall make it easy and attractive.

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II.—TRIGONOMETRIC FUNCTIONS.

III.—PLANE TRIANGLES.

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V. — DERIVATIVES, SERIES, TABLES, ETC.

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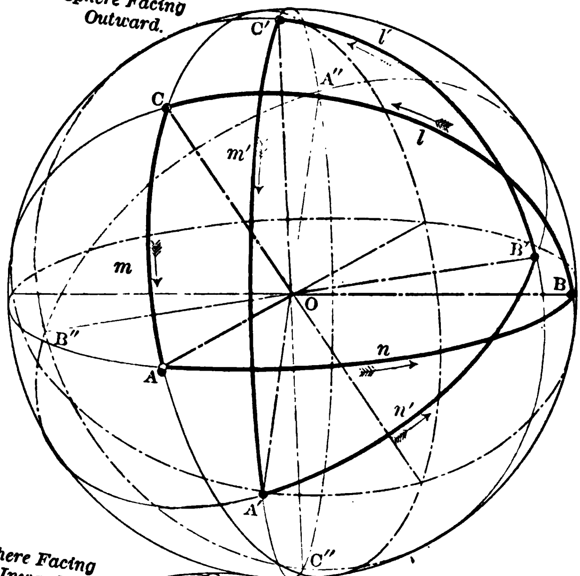

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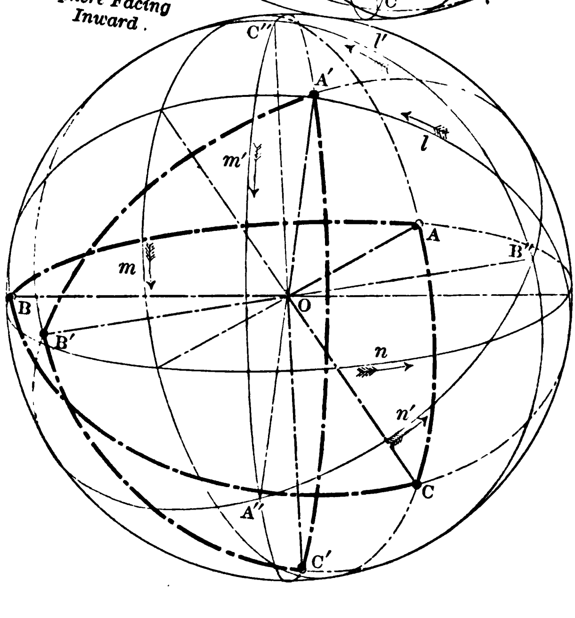


TRIGONOMETRY.

TRIGONOMETRY is that branch of mathematics which treats of the numerical relations of angles and triangles. It is essentially algebraic in character, but is founded on geometry.

I. THE POSITION OF A POINT IN A PLANE.

§ 1. POSITIVE AND NEGATIVE LINES.

A straight line may be generated by a point moving along it in either of two directions. In pure geometry it makes no difference in which of these two ways the point moves; but in the applications of algebra to geometry two separate lines are conceived to be thus generated, lying in the same path, but having opposite directions. Such lines are called *directed lines*, as distinguished from the path itself, which is a *non-directed line.*

If two lines lie in the same path and have $\left\{\begin{array}{l}\text{the same direction}\\ \text{opposite directions}\end{array}\right.$ they are $\left\{\begin{array}{l}\textit{coincident}\\ \textit{opposite}\end{array}\right.$ *lines.*

Two parallel lines have $\left\{\begin{array}{l}\textit{the same direction}\\ \textit{opposite directions}\end{array}\right.$ if their generating points move from an intersecting line upon $\left\{\begin{array}{l}\text{the same side}\\ \text{opposite sides}\end{array}\right.$ of that line.

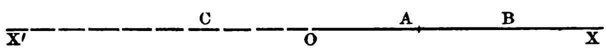
If a segment of a line be designated by two letters, that part of the line is meant that lies between the points marked by the

letters; and the direction of a segment is indicated by the order of the letters, the first letter marking the *initial point* and the other letter the *terminal point*.

Segments of a line reaching $\left\{\begin{matrix}\text{forward, in} \\ \text{backward, against}\end{matrix}\right.$ the direction of the line, are $\left\{\begin{matrix}\textit{positive} \\ \textit{negative}\end{matrix}\right.$ *segments.*

If the position of a single point upon a line be known, that point may be taken as a point of reference, the *origin*, and the positions of other points upon the same line are determined by their distances and directions from this point. The indefinite segment of a line that reaches from the origin $\left\{\begin{matrix}\text{forward} \\ \text{backward}\end{matrix}\right.$ is the $\left\{\begin{matrix}\textit{positive} \\ \textit{negative}\end{matrix}\right.$ *end* of the line.

E.g., if the horizontal line x'ox have the direction ox,

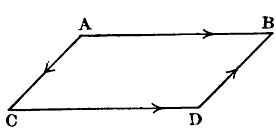


then the segments AB, CA, CB are positive, in this figure,
but BA, AC, BC are negative;
and if O be the origin, $\left\{\begin{matrix}\text{ox} \\ \text{ox}'\end{matrix}\right.$ is the $\left\{\begin{matrix}\text{positive} \\ \text{negative}\end{matrix}\right.$ end of x'ox.

So, A, B, C are determined by the segments OA, OB, OC.

If two segments, not necessarily upon the same line, have the same length and be $\left\{\begin{matrix}\text{both positive or both negative,} \\ \text{the one positive, the other negative,}\end{matrix}\right.$ they are $\left\{\begin{matrix}\textit{equal} \\ \textit{opposite}\end{matrix}\right.$ *segments.*

E.g., in the parallelogram ABCD the segments AB, CD are equal.



So BA, DC, CA, BD are equal;
but AB, DC, CA, DB are opposite.

§ 2. ADDITION OF STRAIGHT LINES.

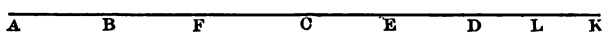
Two or more segments of a straight line are added by placing the initial point of the second upon the terminal of the first, the initial of the third upon the terminal of the second, and so on; and the sum of all the segments so added is the segment that reaches from the first initial to the last terminal point. When a positive segment is added, the terminal point moves forward; when a negative segment is added, it moves backward.

E.g., in the figure below

$$\text{AB} + \text{BA} = 0, \quad \text{AB} + \text{BC} = \text{AC}, \quad \text{AB} + \text{BC} + \text{CA} = 0.$$

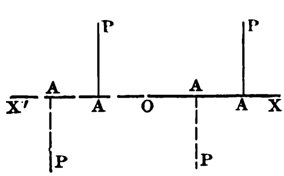
$$\text{AB} + \text{BC} + \text{CD} + \cdots + \text{KL} = \text{AL}.$$

$$\text{AB} + \text{BC} + \text{CD} + \cdots + \text{KL} + \text{LA} = 0.$$



§ 3. THE ABSCISSA AND ORDINATE OF A POINT.—PROJECTIONS.

If OX be a given line in a plane and O a given point upon it, then with OX as the *axis* and O as the *origin*, the position of any other point, P, lying in the plane, is determined by drop-



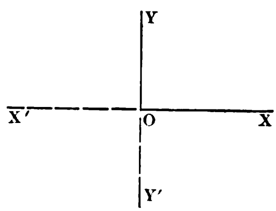
ping a perpendicular, AP, from P on OX and noting the lengths and directions of the lines OA, AP. OA is the *abscissa* and AP the *ordinate* of the point P, as to the axis OX and the origin O.

If the reader so place himself before the figure that OX is horizontal and directed to the right, then by general agreement, unless the contrary be distinctly stated, abscissas reaching to

the $\left\{\begin{matrix}\text{right}\\ \text{left}\end{matrix}\right.$ are $\left\{\begin{matrix}\textit{positive}\\ \textit{negative}\end{matrix}\right.$ *abscissas*, and ordinates reaching $\left\{\begin{matrix}\text{upward}\\ \text{downward}\end{matrix}\right.$ are $\left\{\begin{matrix}\textit{positive}\\ \textit{negative}\end{matrix}\right.$ *ordinates; i.e.*, abscissas are segments of a right-hand line, and ordinates of an upward line.

By a like general agreement the letter x is used as the symbol of the length of an abscissa, and y as that of an ordinate, each measured by its positive unit line; and the two together are called the *rectangular coordinates* of the point. The position of a point is expressed by the equations $x=a$, $y=b$, or simply by the symbol ab.

If the reader look along a straight line in its direction, then a straight line perpendicular to the given line and crossing it from right to left is a *normal* to that line.



If through the origin a line OY be drawn normal to OX, then OY is a secondary axis, the two axes together are the *axes* of *coordinates* (the *x-axis* and the *y-axis*), and they divide the plane into four quarters, whereof XOY is the *first quarter*, YOX′ the *second*, X′OY′ the *third*, Y′OX the *fourth;* and of points in these four quarters the intrinsic signs of the coordinates are:

$$+,+,\qquad -,+,\qquad -,-,\qquad +,-.$$

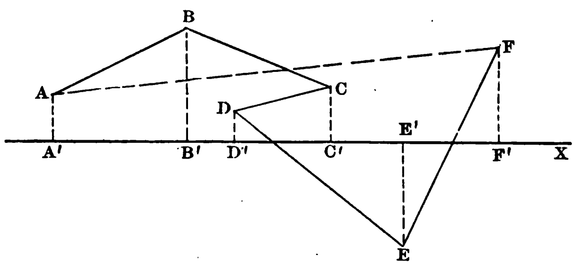
If $x=0$, the point is on the y-axis; if $y=0$, it is on the x-axis; if $x=0$, $y=0$, it is at the origin.

PROJECTIONS.

The *orthogonal projection* of a point on a line is the foot of the perpendicular from the point to the line; and, in this book, by *projection* is always meant orthogonal projection.

The projection of a limited line upon another line is the part of the second line, the *line of projection*, that reaches from the projection of the initial point of the first line to the projection of its terminal point. The projection of a line upon the x-axis is its *x-projection*, that upon the y-axis is its *y-projection*. Projections upon the same line are *like projections*.

If a line be made up of limited straight lines so placed that the terminal point of the first is the initial point of the second, the terminal of the second the initial of the third, and so on, such a line is a *broken line;* and the projection of a broken line upon a straight line is the algebraic sum of the projections of the parts upon that straight line.



E.g., let ABCDEF be any broken line, and OX any straight line; then the projection of ABCDEF on OX is

$$A'B' + B'C' + C'D' + D'E' + E'F'.$$

THEOR. 1. *The projection of a broken line upon a straight line is equal to the like projection of the straight line that reaches from the initial to the terminal point of the broken line.*

E.g., proj. ABCDEF $= A'B' + B'C' + C'D' + E'F' = A'F' =$ proj. AF.

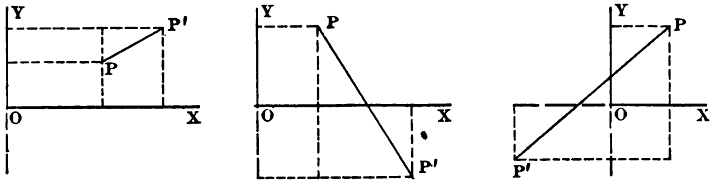
COR. *The projection of a closed polygon upon a straight line is zero.*

THEOR. 2. *The square of the distance between any two points in a plane is the sum of the squares of its projections upon any two rectangular axes.*

For these projections are equal to the sides of a right triangle whose hypothenuse is the given line. [fig. p. 6

COR. *If r be the distance between two points xy, x'y', then*

$$r^2 = (x - x')^2 + (y - y')^2.$$ [geom.



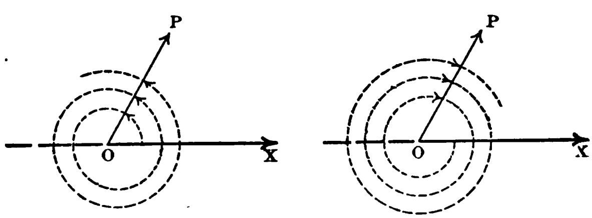
EXAMPLES.

1. Construct the points in a plane whose rectangular coordinates are: 5, 3; 2, 9; 8, ⁻7; ⁻3, 4; ⁻5, ⁻8.
2. Find the ten distances between these five points.
3. Join the five points by a broken line and find the length of its projection on each axis.

§ 4. POSITIVE AND NEGATIVE ANGLES.

If two straight lines coincide, and if one of them swing about a common point as a pivot; then either of the two openings, that between the positive ends of the two lines and that between their negative ends, is the *plane angle* between the lines.

The fixed line is the *initial line*, and the swinging line is the *terminal line*. The swinging line has made one revolution when,



by a continuous forward motion, it has come again to the position it first occupied. An angle is not limited in trigonometry by the words "acute" and "obtuse"; it may exceed a half

revolution, or a whole revolution, or it may be any number of revolutions.

E.g., the angle XOP may be any one of an infinite number of angles got by swinging the line OP continuously about the point O; it gives another of the angles XOP whenever it comes again to a particular position.

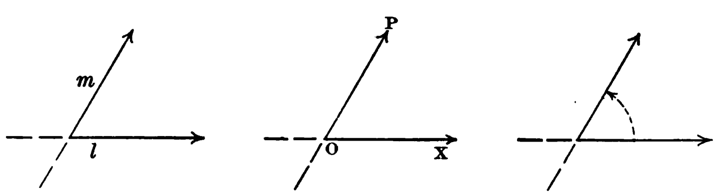
Such returns are *periodic*, and the angles so generated are *congruent angles.*

The angle bounded by the initial line and the first position of the terminal line is the *primary* angle.

An angle is in the first, second, third, or fourth quarter, according as its terminal line lies in the first, second, third, or fourth quarter, counting from the initial line.

But the terminal line may come to any given position either by a clock-wise or by a counter-clock-wise motion, and, by general agreement, if by a clock-wise motion (like the apparent motion of the sun and moon as seen by an observer in the northern hemisphere), the angle generated is a *negative angle;* but if by a counter-clock-wise motion (like the real motions of the planets about the sun), the angle generated is a *positive angle.*

An angle is designated by naming its initial and terminal lines in order, and in a diagram it is shown by a directed arc that subtends it.



E.g., $\angle lm$ is the angle from the line l to the line m.

So, if OX, OP be the initial and terminal lines, the angle is designated by XOP.

Conversely, if an angle be designated by XOP, then OX, OP are the positive ends of the initial and terminal lines.

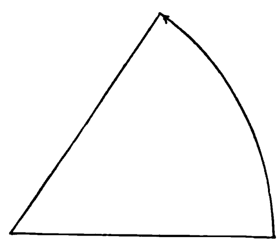
§ 5. MEASUREMENT OF ANGLES.

An entire revolution gives an absolute standard unit for the measurement of angles. This unit is divided into four equal parts, right angles; a right angle into ninety equal parts, *degrees;* a degree into sixty equal parts, *minutes;* a minute into sixty equal parts, *seconds.* Degrees, minutes, and seconds are marked $^\circ$ $'$ $''$.

E.g., an angle of 6 deg. 29 min. 33 sec. is written $6^\circ\ 29'\ 33''$.

This measure is called *degree-measure.*

Another absolute standard unit is the angle at the centre of a circle that is subtended by an arc equal in length to the radius of the circle.



This angle, being subtended by a radius-arc, is called a *radian;* and since the half-circle contains π radius-arcs, a radian is a π^{th} part of two right angles, and is constant. Radians may be marked ^{r}; and this measure will be called *radius-measure.*

In radius-measure an entire revolution is expressed by $2\,\pi^r$, a half revolution by π^r, a right angle by $\frac{1}{2}\,\pi^r$, a third of a right angle by $\frac{1}{6}\,\pi^r$, $\cdots$.

i.e., $\quad \pi^r = 180^\circ,\ \ \frac{1}{2}\,\pi^r = 90^\circ,\ \ \frac{1}{6}\,\pi^r = 30^\circ,\ \ 1^r = 57^\circ\ 17'\ 44''.8.$

So, $\quad 1^\circ = \pi^r : 180 = .0174533^r,\ \ 1' = .0002909^r.$

The index ^{r} may generally be dropped without misunderstanding. It must not be mistaken for an exponent.

Degree-measure is better for the computation of triangles; and, generally, radius-measure is better for theoretical work.

Another measure, used in astronomy and navigation, may be called *hour-measure*. One revolution of the earth on its axis is divided into twenty-four *hours*, each hour into sixty *minutes*, and each minute into sixty *seconds*. Hours, minutes, and seconds of hour-measure are marked $^h\ ^m\ ^s$.

E.g., $1^h = 15°$, $1° = 4^m$, $1^m = 15'$, $1' = 4^s$, $1^s = 15''$.

Another measure, now seldom used, is *grade-measure*. One revolution is divided into four hundred *grades*, each grade into one hundred *minutes*, and each minute into one hundred *seconds*. These grades, minutes, and seconds are marked $^g\ ^{\backprime}\ ^{\backprime\backprime}$.

Another measure, used by mariners, is that of *points* and *quarter-points*. One revolution is divided into thirty-two points; and one point is equal to $11°\ 15'$.

EXAMPLES.

☞ The signs $^+$, $^-$, $^\pm$ are signs of quality, not of operation.
[O. W. J. alg. I § 3

1. Prove that the number of radians in an angle is expressed by the ratio of the arc subtending it to the radius of the circle, *i.e.*, by the number of radii in the arc.

2. Express in degree-measure the angles:
$\frac{1}{5}\pi$, $\frac{1}{7}\pi$, $\frac{1}{9}\pi$, $\frac{5}{3}\pi$, 3.1416^r, $^-.7854^r$, 1^r, 1.5^r, $^-2^r$, $\pi + 1^r$.

3. Express in radius-measure the angles:
$14°$, $15°$, $24°$, $120°$, $137°\ 15'$, $^-4800°$, $13'$, $24''$.

4. If the radius be an inch, find the length of the arcs:
$14°$, $15°$, $120°$, $57°\ 17'\ 44''.8$, $1°$, $\frac{1}{5}\pi$, $\frac{1}{7}\pi$, 2^r, $\pi + 1^r$.
So if the radius be five inches.

5. How many radii in an arc of: $20°$, $180°$, 3^r, 100^g?

6. If the radius be 10 in., find the number of radians subtended by an arc of: 13 in., π in., $10°$, $5'\ 13''$, 3 quadrants, 100^g.

7. The angle 3.42^r is subtended by an arc of 5.71 in., find: the radius; the arcs opposite $\frac{1}{2}\pi^r$, 1^r, $5°$; the angles in radians and in degrees opposite a one-inch arc, a two-radius arc, .7854 radius-arc, 5 quadrants, 25^g.

8. If the circumference of a circle be 30 in., find the arcs opposite π^r, $30°$, 3^r, 20^g.

9. How many radians and how many degrees are subtended by: $2\frac{1}{2}$ radius-arcs, π radius-arcs, $3\frac{1}{2}$ quadrants?

10. How many radians in $17° 13' 15''$, $10°$, 200^g?

11. Convert into grades, minutes, and seconds: 1^r, $86° 13' 17''$.

13. An angle of three radians at the centre of a sphere subtends a two-foot arc of a great circle; find the radius.

§ 6. ADDITION OF ANGLES.

Two or more angles that lie in the same plane are added by placing the initial line of the second angle upon the terminal of the first, the initial of the third upon the terminal of the second, and so on; and the sum of all the angles so added is the angle bounded by the first initial and the last terminal line. When a positive angle is added, the terminal line swings forward; when a negative angle is added, it swings backward.

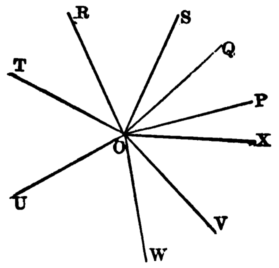
E.g., in the figure below:

$$\text{XOP} + \text{POX} = \text{XOX}, \qquad \text{XOP} + \text{POQ} = \text{XOQ},$$

$$\text{XOP} + \text{POQ} + \text{QOX} = \text{XOX},$$

$$\text{XOP} + \text{POQ} + \text{QOR} + \cdots + \text{VOW} = \text{XOW},$$

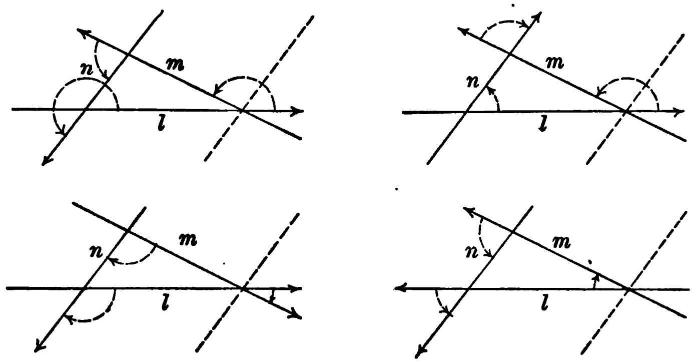
$$\text{XOP} + \text{POQ} + \text{QOR} + \cdots + \text{VOW} + \text{WOX} = \text{XOX}.$$



And this is true whether the angles be positive or negative.

E.g., XOP + POQ is always one of the congruent angles XOQ.

So, if l, m, n be any three lines in a plane,



then $\qquad \angle lm + \angle mn = \angle ln.$

So, if $l, m, n, \cdots r, s$ be any lines in a plane,

then $\qquad \angle lm + \angle mn + \cdots + \angle rs = \angle ls$

and $\qquad \angle lm + \angle mn + \cdots + \angle rs + \angle sl = \angle ll = 2\, n\, \pi,$

wherein n may be 0 or any other integer, positive or negative.

To subtract an angle is to add its opposite.

§ 7. COMPLEMENT AND SUPPLEMENT OF AN ANGLE.

The *complement* of an angle is its defect from a right angle; *i.e.*, it is the remainder when from a right angle the given angle is subtracted.

The *supplement* of an angle is its defect from two right angles; *i.e.*, it is the remainder when from two right angles the given angle is subtracted.

The complement of a positive angle less than a right angle is a positive angle less than a right angle; of a positive angle greater than a right angle, a negative angle; of a negative angle, a positive angle greater than a right angle.

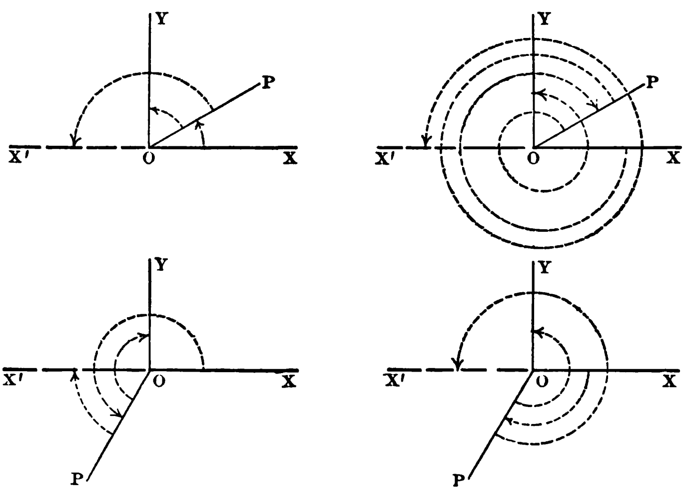
So, the supplement of a positive angle less than two right angles is a positive angle less than two right angles; of a posi-

tive angle greater than two right angles, a negative angle; of a negative angle, a positive angle greater than two right angles.

E.g., the complement of $75°$ is $15°$; of $100°$ is $^{-}10°$;
of $^{-}10°$ is $100°$; of $\frac{1}{6}\pi$ is $\frac{1}{3}\pi$; of 2π is $^{-}\frac{3}{2}\pi$.

So, the supplement of $75°$ is $105°$; of $200°$ is $^{-}20°$;
of $^{-}20°$ is $200°$; of $\frac{1}{6}\pi$ is $\frac{5}{6}\pi$; of 2π is $^{-}\pi$.

In the figure below, XOY is a right angle, XOX′ two right angles, XOP any angle, POY its complement, POX′ its supplement.



Two angles are $\left\{\begin{array}{l}\textit{complementary}\\ \textit{supplementary}\end{array}\right.$ when their algebraic sum is $\left\{\begin{array}{l}\text{one right angle.}\\ \text{two right angles.}\end{array}\right.$

EXAMPLES.

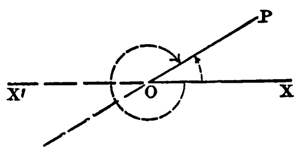
1. Find the complements and the supplements of the angles:
   $39°$, $215°$, $107° 12' 15''$, $^{-}36° 12'$, $\frac{1}{5}\pi$, $\frac{6}{7}\pi$, $\frac{11}{9}\pi$, $^{-}\frac{5}{3}\pi$.

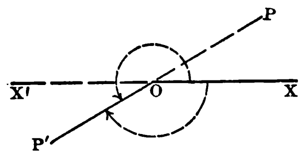
2. Write down the complement of $\frac{2}{3}\pi$, and the supplement of $\frac{5}{6}\pi$, in radians, degrees, and grades.

3. Find the angle, the supplement of whose complement is $150°$.

§ 8. THE BEARING AND DISTANCE OF A POINT.

If OX be a given line in a plane and O a given point upon it; then any other point P lying in the plane is determined by joining OP, and noting the size and sign of the angle XOP and the length of the line OP.





XOP is the *vectorial angle* or *bearing*, and OP is the *radius vector* or *distance* of the point P, as to the axis OX and the origin O.

The bearing XOP and the distance OP, taken together, are the *polar coordinates* of the point P.

But the position OP may be reached by turning through either of the positive or either of the negative angles XOP.

The point P may also be determined by measuring either of the positive or either of the negative angles XOP′ and the negative distance OP; so that for every point there are four pairs of polar coordinates wherein the vectorial angle lies between ${}^{-}360°$ and ${}^{+}360°$:

$$\angle^{+}\text{XOP}, {}^{+}\text{OP};\ \angle^{-}\text{XOP}, {}^{+}\text{OP};\ \angle^{+}\text{XOP}', {}^{-}\text{OP};\ \angle^{-}\text{XOP}', {}^{-}\text{OP}.$$

E.g., the same point is determined by the coordinates:

$${}^{+}60°, {}^{+}10;\quad {}^{-}300°, {}^{+}10;\quad {}^{+}240°, {}^{-}10;\quad {}^{-}120°, {}^{-}10.$$

EXAMPLES.

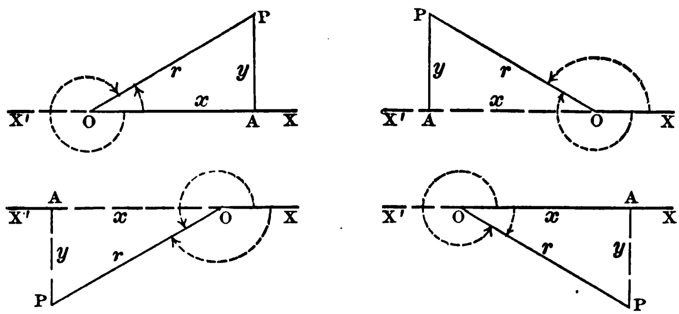
Construct the points whose polar coordinates are:

$30°, 10$; ${}^{-}330°, 10$; $30°, {}^{-}10$; $210°, 10$; ${}^{-}150°, 10$; ${}^{-}330°, {}^{-}10$; $\pi, 8$; $\pi, {}^{-}8$; ${}^{-}\pi, 8$; ${}^{-}\pi, {}^{-}8$; $0°, 8$; $0°, {}^{-}8$; $\frac{5}{3}\pi, 4$; $\frac{5}{3}\pi, {}^{-}4$; ${}^{-}\frac{1}{3}\pi, 4$; ${}^{-}\frac{1}{3}\pi, {}^{-}4$; $\frac{2}{3}\pi, 4$; ${}^{-}\frac{2}{3}\pi, {}^{-}4$; $\frac{8}{3}\pi, {}^{-}4$.

II. TRIGONOMETRIC FUNCTIONS.

§ 1. FUNCTIONS OF A SINGLE ANGLE.

Let XOP be any angle, positive or negative, P any point upon the terminal line, and AP ordinate of P as to OX;



then OP, OA, AP are the distance, abscissa, and ordinate of P; and the six ratios of these three lines are the six principal *trigonometric functions* of the angle XOP, viz.:

the ratio	*i.e.*,	is the	written
ordinate : distance	$y : r$	*sine* of XOP	sin XOP,
abscissa : distance	$x : r$	*cosine* of XOP	cos XOP,
ordinate : abscissa	$y : x$	*tangent* of XOP	tan XOP,
abscissa : ordinate	$x : y$	*cotangent* of XOP	cot XOP,
distance : abscissa	$r : x$	*secant* of XOP	sec XOP,
distance : ordinate	$r : y$	*cosecant* of XOP	csc XOP.

The *versed sine* and *coversed sine* are defined by the equations: vers XOP $= 1 -$ cos XOP, covers XOP $= 1 -$ sin XOP.

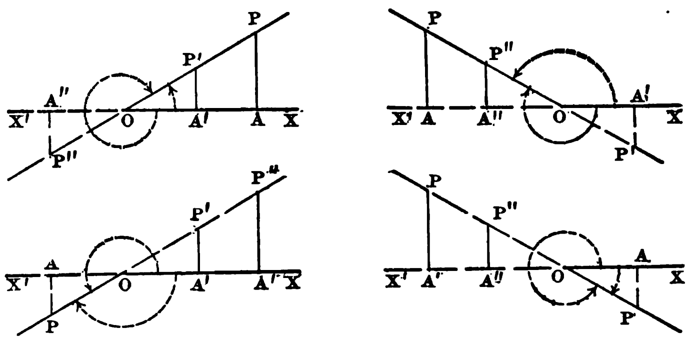
The expressions $\sin^{-1}a$, $\cos^{-1}a$, $\tan^{-1}a$, $\cdots$ are *anti-functions*, and are read the *anti-sine* of a, the *anti-cosine* of a, $\cdots$; they mean the angle whose sine is a, the angle whose cosine is a, $\cdots$.

E.g., if $a = \sin\theta$, then $\theta = \sin^{-1}a$;
if $b = \cos\theta$, then $\theta = \cos^{-1}b$.

CONSTANCY OF FUNCTIONS.

THEOR. 1. *For a given angle the functions are constant.*

For, let XOP be any angle, P, P′, P″, ⋯ any points on OP; and let AP, A′P′, A″P″, ⋯ be ordinates of P, P′, P″, ⋯ as to OX;



then ∵ the triangles OAP, OA′P′, OA″P″, ⋯ are similar, [geom.

and like coordinates of points upon $\left\{\begin{matrix}\text{the same}\\ \text{opposite}\end{matrix}\right.$ sides of O have $\left\{\begin{matrix}\text{the same}\\ \text{opposite}\end{matrix}\right.$ signs, and so for the distances;

∴ like ratios are equal, both in magnitude and in quality;

i.e., $y : r = y' : r' = y'' : r'' = \cdots,$

and so for the other functions. Q. E. D.

PERIODICITY OF FUNCTIONS.

THEOR. 2. *The like functions of congruent angles are identical.*

For, let θ be any plane angle;

then ∵ $^{\pm}2\pi$, $^{\pm}4\pi$, $\cdots {}^{\pm}2n\pi$ stand for one, two, ⋯ n entire revolutions, forward or backward,

∴ the terminal line has the same position for the angles

$$\theta,\ {}^{\pm}2\pi+\theta,\ {}^{\pm}4\pi+\theta,\ \cdots {}^{\pm}2n\pi+\theta,$$

and the r's, the x's, the y's may be made identical, each with each, for all the angles thus formed;

$\therefore$ the ratios are identical, sine with sine, cosine with cosine, and so on. Q. E. D.

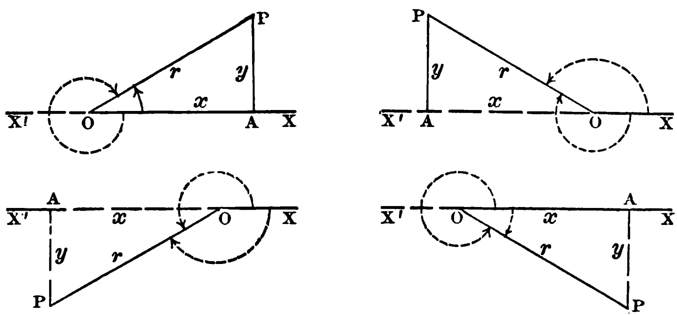
POSITIVE AND NEGATIVE FUNCTIONS.

THEOR. 3. *For angles in the first quarter all the trigonometric functions are positive.*

In the second quarter, the sine and cosecant are positive; the cosine, secant, tangent, and cotangent are negative.

In the third quarter the tangent and cotangent are positive; the sine, cosecant, cosine, and secant are negative.

In the fourth quarter the cosine and secant are positive; the sine, cosecant, tangent, and cotangent are negative.



For, if r be taken positive;

then $\because$ in the first quarter r, x, y are all positive, [df.
in the second quarter r, y are positive, and x is negative,
in the third quarter r is positive, and x, y are negative,
in the fourth quarter r, x are positive, and y is negative,

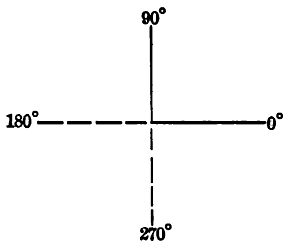
$\therefore$ the qualities of the ratios are as given above. Q. E. D.

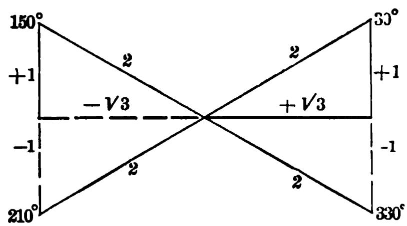
And so they are if r be taken negative; the reader may prove.

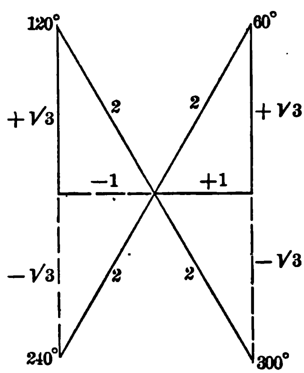
EXAMPLES.

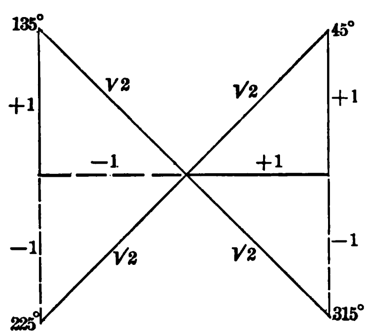
By direct reference to the definitions and by aid of the figures :

1. Tabulate the functions of $\frac{1}{2}n\pi$, *i.e.*, of 0°, 90°, 180°, ⋯.
2. Tabulate the functions of $(\frac{1}{2}n \pm \frac{1}{6})\pi$, *i.e.*, of 30°, 60°, ⋯.
3. Tabulate the functions of $(n \pm \frac{1}{4})\pi$, *i.e.*, of 45°, 135°, ⋯.
4. Find sin 225°, $^-585°$, 810°, $^-960°$, 3π, $^-\frac{27}{4}\pi$.
5. Find cos 315°, $^-675°$, 960°, $^-1110°$, $\frac{23}{6}\pi$, $^-8\pi$.
6. Find tan 495°, $^-945°$, 1110°, $^-1260°$, $\frac{28}{6}\pi$, $^-\frac{37}{4}\pi$.
7. Find cot 675°, $^-1035°$, 1260°, $^-1410°$, $\frac{33}{6}\pi$, $^-\frac{42}{4}\pi$.
8. Find sec 855°, $^-1215°$, 1410°, $^-1560°$, $\frac{38}{6}\pi$, $^-\frac{47}{4}\pi$.
9. Find csc 1035°, $^-1395°$, 1560°, $^-1710°$, $\frac{43}{6}\pi$, $^-13\pi$.







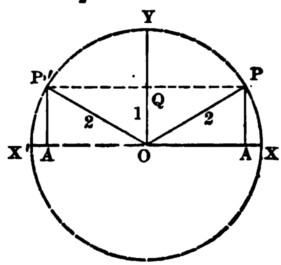


10. Find $\sin^{-1} {}^{\pm}\frac{1}{2}\sqrt{3}$, $\sin^{-1}\cos 50°$; $\cos^{-1} {}^{\pm}\sqrt{\frac{1}{2}}$, $\cos^{-1}\sin 50°$.
11. Find $\tan^{-1} {}^{\pm}1$, $\tan^{-1}\tan\frac{1}{8}\pi$; $\cot^{-1}0$, $\cot^{-1}\cot\frac{3}{8}\pi$.

12. Find $\sec^{-1}\infty$, $\sec^{-1}\sec\frac{5}{8}\pi$; $\csc^{-1}\infty$, $\csc^{-1}\csc\frac{7}{8}\pi$.

13. Construct $\sin^{-1}\frac{1}{2}$, $^{-}\frac{2}{3}$, $\frac{3}{5}$, 1, 0; $\cos^{-1}\frac{3}{5}$, $^{\pm}\frac{1}{2}$, $^{-}1$;
$\tan^{-1}\frac{1}{3}$, $\frac{3}{4}$, 0, $^{-}1$, ∞; $\cot^{-1}\frac{1}{4}$, $^{-}\frac{3}{4}$, $^{\pm}1$.

E.g., to construct $\sin^{-1}\frac{1}{2}$:



with radius 2 draw a circle about O as centre;
on the axis OY take Q such that OQ is 1;
through Q draw a parallel to OX, cutting the circle in P, P';
then ∵ the ordinates of P, P' are 1, and their distances 2,
∴ of XOP, XOP', and their congruents, the sines are $\frac{1}{2}$,
and there are no other such angles.

Make a like construction, with given abscissas and distances, for the anti-cosines;
so with given ordinates and abscissas for the anti-tangents and anti-cotangents.

There are two primary angles for each function.

14. Show, by geometry, that $\sin 2\theta < 2\sin\theta$, if $0 < \theta < \pi$.

15. Divide an angle into two parts that shall have their sines in a given ratio, their cosines in a given ratio, ….

16. Construct an angle whose tangent is four times its sine.

17. If $\frac{1}{4}\pi < \theta < \frac{1}{2}\pi$, show that $\tan\theta > \cot\theta$.

18. Find the primary values for α, β that satisfy the equations $\sin(3\alpha - 2\beta) = 1$, $\sin(4\beta - \alpha) = \frac{1}{2}$.

19. If the radius of a circle be divided in extreme and mean ratio the greater segment is the side of a regular inscribed decagon: hence find the functions of 18° and of 36°.

§ 2. RELATIONS OF FUNCTIONS OF A SINGLE ANGLE.

THEOR. 4. *If θ be any plane angle, then:*

$$\sin\theta\cdot\csc\theta=1,\quad \cos\theta\cdot\sec\theta=1,\quad \tan\theta\cdot\cot\theta=1;$$
$$\tan\theta=\sin\theta:\cos\theta,\quad \cot\theta=\cos\theta:\sin\theta;$$
$$\sin^2\theta+\cos^2\theta=1,\ 1+\tan^2\theta=\sec^2\theta,\ 1+\cot^2\theta=\csc^2\theta.$$

For $\because$ $\sin\theta=y:r,\quad \cos\theta=x:r,\quad \tan\theta=y:x,$

$\csc\theta=r:y,\quad \sec\theta=r:x,\quad \cot\theta=x:y,$ [df.

$\therefore$ $\sin\theta\cdot\csc\theta=1,\quad \cos\theta\cdot\sec\theta=1,\quad \tan\theta\cdot\cot\theta=1;$

and $\tan\theta=\sin\theta:\cos\theta,\quad \cot\theta=\cos\theta:\sin\theta.$ Q. E. D.

So $\because$ $x^2+y^2=r^2,$ [geom.

$\therefore$ $\frac{x^2}{r^2}+\frac{y^2}{r^2}=1,\qquad 1+\frac{y^2}{x^2}=\frac{r^2}{x^2},\qquad \frac{x^2}{y^2}+1=\frac{r^2}{y^2},$

i.e., $\cos^2\theta+\sin^2\theta=1,\ 1+\tan^2\theta=\sec^2\theta,\ \cot^2\theta+1=\csc^2\theta.$

COR. *If θ be any plane angle, then:*

$\sin\theta=$	$\cos\theta=$	$\tan\theta=$	$\cot\theta=$	$\sec\theta=$	$\csc\theta=$
$\sin\theta$	$\sqrt{1-\sin^2\theta}$	$\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$	$\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$	$\frac{1}{\sqrt{1-\sin^2\theta}}$	$\frac{1}{\sin\theta}$
$\sqrt{1-\cos^2\theta}$	$\cos\theta$	$\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$	$\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}$	$\frac{1}{\cos\theta}$	$\frac{1}{\sqrt{1-\cos^2\theta}}$
$\frac{\tan\theta}{\sqrt{\tan^2\theta+1}}$	$\frac{1}{\sqrt{\tan^2\theta+1}}$	$\tan\theta$	$\frac{1}{\tan\theta}$	$\sqrt{\tan^2\theta+1}$	$\frac{\sqrt{\tan^2\theta+1}}{\tan\theta}$
$\frac{1}{\sqrt{\cot^2\theta+1}}$	$\frac{\cot\theta}{\sqrt{\cot^2\theta+1}}$	$\frac{1}{\cot\theta}$	$\cot\theta$	$\frac{\sqrt{\cot^2\theta+1}}{\cot\theta}$	$\sqrt{\cot^2\theta+1}$
$\frac{\sqrt{\sec^2\theta-1}}{\sec\theta}$	$\frac{1}{\sec\theta}$	$\sqrt{\sec^2\theta-1}$	$\frac{1}{\sqrt{\sec^2\theta-1}}$	$\sec\theta$	$\frac{\sec\theta}{\sqrt{\sec^2\theta-1}}$
$\frac{1}{\csc\theta}$	$\frac{\sqrt{\csc^2\theta-1}}{\csc\theta}$	$\frac{1}{\sqrt{\csc^2\theta-1}}$	$\sqrt{\csc^2\theta-1}$	$\frac{\csc\theta}{\sqrt{\csc^2\theta-1}}$	$\csc\theta$

E.g., $\because$ $\sec\theta=\sqrt{(1+\tan^2\theta)},$

$\therefore$ $\cos\theta=1:\sec\theta=1:\sqrt{(1+\tan^2\theta)},$

and $\sin\theta=\tan\theta\cos\theta=\tan\theta:\sqrt{(1+\tan^2\theta)}.$

For a given $\left\{\begin{matrix}\text{sine}\\ \text{cosecant}\end{matrix}\right.$ the $\left\{\begin{matrix}\text{cosecant}\\ \text{sine}\end{matrix}\right.$ has but one value;

for a given $\left\{\begin{matrix}\text{cosine}\\ \text{secant}\end{matrix}\right.$ the $\left\{\begin{matrix}\text{secant}\\ \text{cosine}\end{matrix}\right.$ has but one value;

for a given $\left\{\begin{matrix}\text{tangent}\\ \text{cotangent}\end{matrix}\right.$ the $\left\{\begin{matrix}\text{cotangent}\\ \text{tangent}\end{matrix}\right.$ has but one value;

but in every other case there are two corresponding values, opposites; as appears alike from the double signs implied in the radicals, and from the relations of the abscissas and ordinates of points in the several quarters; for, with a given distance,
to every abscissa correspond two opposite ordinates,
  and to every cosine correspond two opposite sines;
to every ordinate correspond two opposite abscissas,
  and to every sine correspond two opposite cosines.

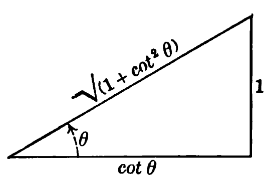
So, with a given abscissa and ordinate there are two opposite distances belonging to two angles that differ by two right angles.

The formulæ, taken two and two, are symmetric.

E.g., those for sine, in terms of cosine, tangent, secant, ···
  with those for cosine, in terms of sine, cotangent, ··· ;
those for tangent, in terms of sine, cosine, secant, ···
  with those for cotangent, in terms of cosine, sine, ··· ;
those for secant, in terms of sine, cosine, tangent, ···
  with those for cosecant, in terms of cosine, sine, ··· ;

The formulæ may also be proved by a geometric method.

E.g., if $\cot\theta$ be given: take as linear unit the consequent of the ratio $\cot\theta$, (abscissa : ordinate);

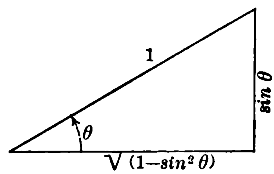


then $\because$ $\cot\theta$ is the length of the antecedent (abscissa)

and $\sqrt{(1+\cot^2\theta)}$ is that of the distance, [geom.

$\therefore\ \sin\theta = 1 : \sqrt{(1+\cot^2\theta)},\quad \cos\theta = \cot\theta : \sqrt{(1+\cot^2\theta)}\ \cdots.$

So, if $\sin\theta$ be given: take as linear unit the consequent of the ratio $\sin\theta$, (ordinate : distance);



then $\because$ $\sin\theta$ is the length of the ordinate

and $\sqrt{(1-\sin^2\theta)}$ is that of the abscissa,

$$\therefore \tan\theta = \sin\theta : \sqrt{(1-\sin^2\theta)},\ \sec\theta = 1 : \sqrt{(1-\sin^2\theta)},\ \cdots.$$

To make the proof general by this method, the reader must draw figures in all four quarters, and verify the formulæ for each figure. They are always true; but the reader cannot know this until he sees that the algebraic processes are independent of the quarters in which the figures lie.

EXAMPLES.

1...7. Prove the identities:

1. $(\sin\alpha\cos\beta - \cos\alpha\sin\beta)^2 + (\cos\alpha\cos\beta + \sin\alpha\sin\beta)^2 = 1.$
2. $\sin^2\alpha\cos^2\beta\,(1+\cot^2\alpha)(1+\tan^2\beta) = 1.$
3. $\sin^4\theta + \cos^4\theta = 1 - 2\sin^2\theta\cos^2\theta.$
4. $\sin^3\theta + \cos^3\theta = (\sin\theta + \cos\theta)(1 - \sin\theta\cos\theta).$
5. $\sec^3\theta\csc^3\theta - 3\sec\theta\csc\theta = \tan^3\theta + \cot^3\theta.$
6. $(\sin\theta + \cos\theta)^3 + (\sin\theta - \cos\theta)^3 = 2\sin\theta\,(3 - 2\sin^2\theta).$
7. $\sin^2\theta = 2\,\mathrm{vers}\,\theta - \mathrm{vers}^2\theta.$

8...12. Solve the equations:

8. $\sin\theta + \cos\theta = 1.$
9. $3\sin\theta + 4\cos\theta = 5.$
10. $\tan\theta + \cot\theta = 2.$
11. $\cot\theta = 2\cos\theta.$

12. $2 \sin^2\theta - 5 \cos\theta = 4.$

13. If $\cos\theta = \tan\theta$, find $\sin\theta$.

14. If $\sin\theta \cos\theta = m$, find $\sin\theta$.

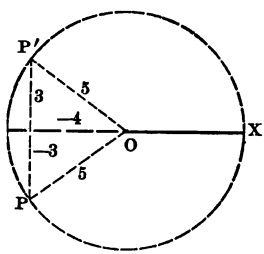
15. Find the remaining functions of θ:

if $\sin\theta = \frac{3}{5},\ \frac{8}{17},\ \frac{3}{4}$; if $\cos\theta = \frac{1}{2},\ {}^-\frac{1}{2}\sqrt{3},\ \frac{12}{13}$;
if $\tan\theta = \frac{3}{4},\ {}^-\frac{1}{3},\ {}^-\frac{2}{3}$; if $\cot\theta = \frac{1}{4},\ \frac{3}{4},\ {}^-a : b$;
if $\sec\theta = \frac{5}{4},\ \frac{85}{77},\ {}^-\frac{65}{16}$; if $\csc\theta = {}^-2,\ \frac{5}{3},\ a : b$.

Either solve by direct reference to the formulæ, or, better, draw a right triangle two of whose lines are in the given ratio, compute the third line, and write down the other ratios.

E.g., if $\cos\theta = {}^-4 : 5$;

with radius 5 and o as centre draw a circle, and with abscissa $^-4$



draw a perpendicular cutting the circle in P, P′, whose ordinates are the two values of $\sqrt{(5^2 - 4^2)}$, *i.e.*, $^+3$, $^-3$;

then the ratios sought are:

$$\sin\theta = {}^\pm 3 : 5,\quad \tan\theta = {}^\pm 3 : 4,\quad \sec\theta = {}^- 5 : 4,$$
$$\csc\theta = {}^\pm 5 : 3,\quad \cot\theta = {}^\pm 4 : 3.$$

16. If $\tan\theta + \cot\theta = m$, find all the functions of θ.

17. If $\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$, find the value of $\frac{a}{\cos\theta} + \frac{b}{\sin\theta}$.

18. Eliminate θ from the equations:

$$\sin\theta + \cos\theta = a, \qquad \tan\theta + \sec\theta = b.$$

and find the relation between a and b.

19. Eliminate ϕ from the equations:

$$\csc\phi - \sin\phi = m, \qquad \sec\phi - \cos\phi = n.$$

20. Eliminate θ, ϕ from the equations: $a\tan\theta = b\tan\phi$,

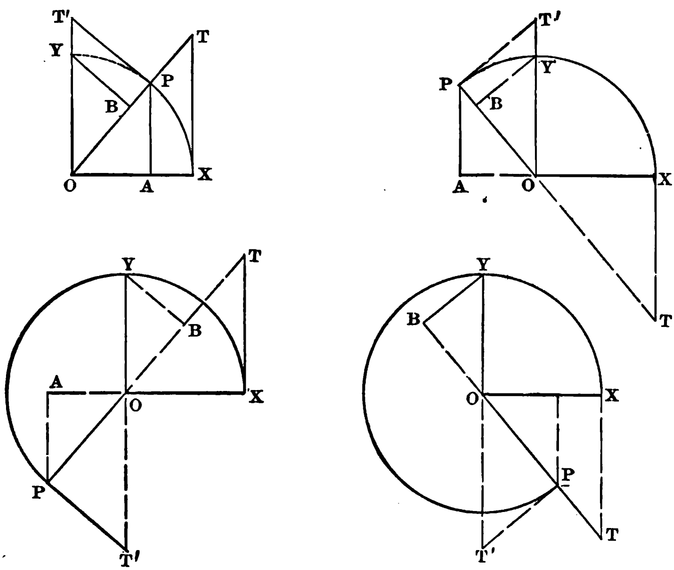
$$a\sin^2\theta + b\cos^2\theta = m, \qquad b\sin^2\phi + a\cos^2\phi = n.$$

21. If $\tan\phi + \sec\phi = a$, find $\sin\phi$.

§ 3. GRAPHIC REPRESENTATION OF FUNCTIONS.

LINE-FUNCTIONS.

Let XP be any arc with centre O and radius OX, and PY the complement of XP;



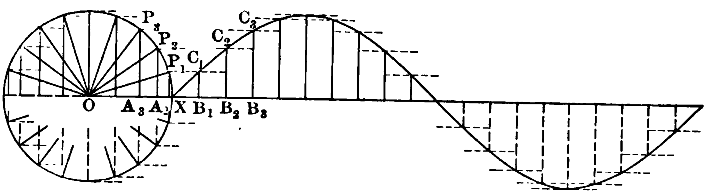
through P, X draw AP, XT ordinates as to OX, and through Y, P draw BY, PT′ ordinates as to OP, with T, T′ on OP, OY;

then AP, OA, XT, OT are the *sine*, *cosine*, *tangent*, *secant* of the arc XP, and BY, OB, PT′, OT′ are the like functions of the complementary arc PY and the *cosine*, *sine*, *cotangent*, *cosecant* of the arc XP.

These lines are called *line-functions* of arcs as distinguished from the *ratio-functions* of angles; and if they be divided by the radius, the ratios so found are the ratio-functions heretofore defined. With arcs of the same radius the ratios of their line-functions are equal to the ratios of the like ratio-functions of their angles.

CURVE OF SINES.

Let OX be the radius of a circle, and divide the circumference into any convenient parts at P_1, P_2, $\cdots$;



draw A_1P_1, A_2P_2, $\cdots$, ordinates of P_1, P_2 $\cdots$ as to OX, and sines of the arcs XP_1, XP_2, $\cdots$;

upon OX lay off XB_1, XB_2, $\cdots$ equal to the arcs XP_1, XP_2, $\cdots$;

at B_1, B_2, $\cdots$ erect perpendiculars to OX and take C_1, C_2, $\cdots$ such that $B_1C_1 = A_1P_1$, $B_2C_2 = A_2P_2$ $\cdots$;

through C_1, C_2, $\cdots$ draw a smooth curve; it is the *curve of sines*, and the following relations are manifest:

The sine is 0 for the angle 0; is nearly as long as the arc for a small angle; increases more and more slowly; is equal to the radius and its ratio is $^+1$, its maximum, for a right angle; decreases, at first very slowly, but faster and faster as the angle approaches two right angles; is 0 for two right angles; decreases from 0 to the opposite of the radius and its ratio is $^-1$, its minimum, as the angle grows from two right angles to three; increases to 0 as the angle grows from three right angles to four; is again 0 at the end of the first revolution. It has all values between the radius and its opposite.

If the revolution be continuous, the values of the sine are periodic, every successive revolution indicating a new cycle, and a new wave in the curve. The sines are equal for pairs of angles symmetric about the y-axis.

OTHER TRIGONOMETRIC CURVES.

The tangent is 0 for the angle 0; increases through the first quarter to $^+\infty$; leaps to $^-\infty$; increases through the second quarter to 0; increases through the third quarter to $^+\infty$; leaps to $^-\infty$; increases through the fourth quarter to 0; and so on. It has all values from $^-\infty$ to $^+\infty$. The tangents are equal for pairs of angles that differ by a half-revolution.

The secant is equal to the radius and its ratio is $^+1$ for the angle 0; increases through the first quarter to $^+\infty$; leaps to $^-\infty$; increases through the second quarter to the opposite of the radius and its ratio is $^-1$; decreases through the third quarter to $^-\infty$; leaps to $^+\infty$; decreases through the fourth quarter to the value at beginning; and so on. It has no value smaller than the radius. The secants are equal for pairs of angles symmetric about the x-axis.

The cosine, cotangent, cosecant have the same bounds as the sine, tangent, secant; they go through like changes and are represented by like curves; but they begin, for the angle 0, with different values, viz.: the radius, ∞, ∞.

EXAMPLES.

1. Show directly from the definitions what are the largest and what the smallest values that each function may have, and state for what angles the several functions take these values.

2. Draw the curve of tangents, curve of secants, curve of cosines, curve of cotangents, and curve of cosecants.

3. Trace the changes, when θ increases from 0 to 2π, in:

$$\sin\theta + \cos\theta, \qquad \tan\theta + \cot\theta, \qquad \sin\theta + \csc\theta,$$
$$\sin\theta - \cos\theta, \qquad \tan\theta - \cot\theta, \qquad \sin\theta - \csc\theta.$$

§ 4. FUNCTIONS OF RELATED ANGLES.

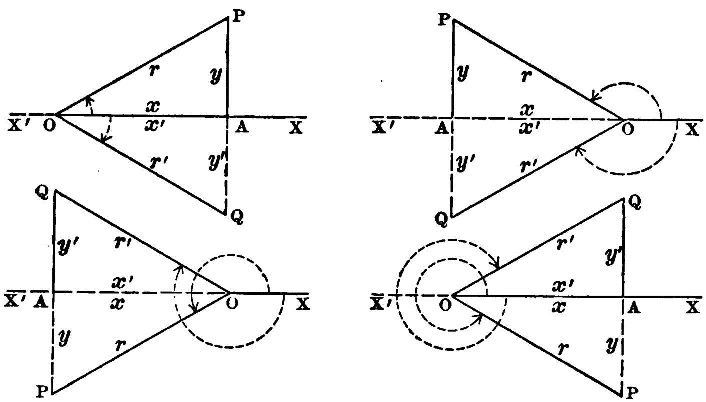
In the theorems that follow, the reader may examine the different figures and apply the proofs to each of them; he will find no angle in any quarter to which they do not apply, and thus he will see that they are general. In particular, he may note the qualities of the lines for angles in the different quarters.

FUNCTIONS OF NEGATIVE ANGLES.

THEOR. 5. *If θ be any plane angle, then:*

$$\sin \overline{-\theta} = -\sin\theta,\ \cos\overline{-\theta} = +\cos\theta,\ \tan\overline{-\theta} = -\tan\theta,$$
$$\csc\overline{-\theta} = -\csc\theta,\ \sec\overline{-\theta} = +\sec\theta,\ \cot\overline{-\theta} = -\cot\theta.$$

For, let XOP be any plane angle θ, and take Q symmetric with P as to OX;



then $\because$ XOQ $= -$ XOP $= -\theta$,

and the distances of P, Q are equal, their abscissas identical, and their ordinates opposite; [geom.

i.e., $\because\ r' = r,\ x' = x,\ y' = -y$, always,

$\therefore\ y' : r' = -y : r,\qquad x' : r' = x : r,\qquad y' : x' = -y : x,$

i.e., $\sin\overline{-\theta} = -\sin\theta,\quad \cos\overline{-\theta} = \cos\theta,\quad \tan\overline{-\theta} = -\tan\theta,$

and so for the other functions. Q. E. D.

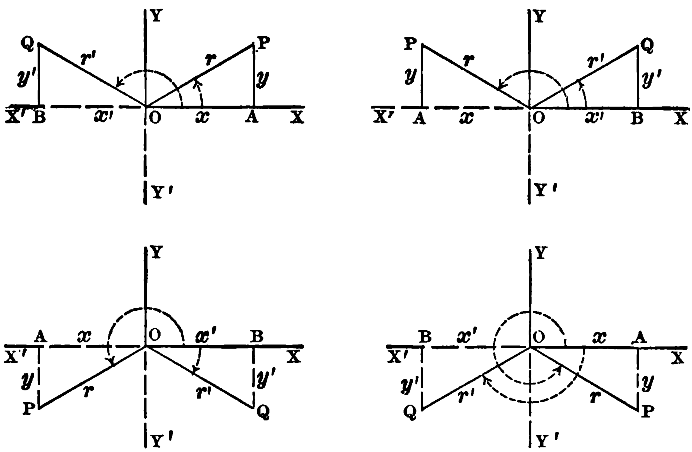
FUNCTIONS OF THE SUPPLEMENT OF AN ANGLE.

THEOR. 6. *If θ be any plane angle, then:*

$$\sin \overline{\pi-\theta} = +\sin\theta,\quad \cos \overline{\pi-\theta} = -\cos\theta,\quad \tan \overline{\pi-\theta} = -\tan\theta,$$

$$\csc \overline{\pi-\theta} = +\csc\theta,\quad \sec \overline{\pi-\theta} = -\sec\theta,\quad \cot \overline{\pi-\theta} = -\cot\theta.$$

For, let XOP be any plane angle θ, and take Q symmetric with P as to OY;



then $\because$ XOQ $= \pi - \theta$,

and the distances of P, Q are equal, their abscissas opposite, and ordinates equal,

i.e., $\because$ $r' = r$, $x' = -x$, $y' = y$, always,

$\therefore$ $y' : r' = y : r$, $\quad x' : r' = -x : r$, $\quad y' : x' = y : -x$,

i.e., $\sin \overline{\pi-\theta} = \sin\theta$, $\cos \overline{\pi-\theta} = -\cos\theta$, $\tan \overline{\pi-\theta} = -\tan\theta$,

and so for the other functions. Q.E.D.

NOTE. If from some point on OX′ a perpendicular fall on OP, the theorem may be proved directly from the supplementary angle POX′.

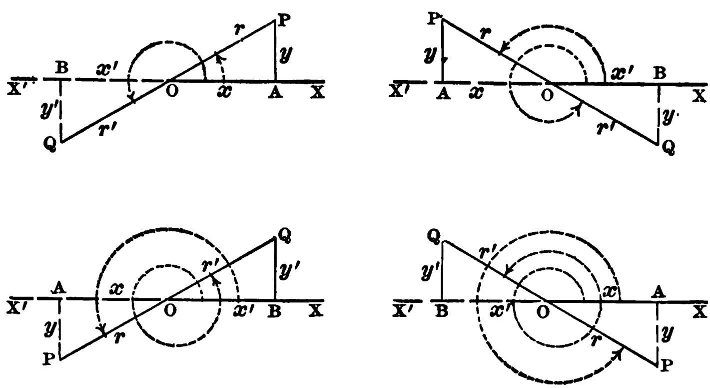
FUNCTIONS OF $\pi + \theta$.

THEOR. 7. *If θ be any plane angle, then:*

$$\sin \overline{\pi + \theta} = -\sin\theta,\ \cos \overline{\pi + \theta} = -\cos\theta,\ \tan \overline{\pi + \theta} = +\tan\theta,$$

$$\csc \overline{\pi + \theta} = -\csc\theta,\ \sec \overline{\pi + \theta} = -\sec\theta,\ \cot \overline{\pi + \theta} = +\cot\theta.$$

For, let XOP be any plane angle θ, and take Q symmetric with P as to the centre O;



then $\because$ XOQ $= \pi + \theta$,

and the distances of P, Q are equal, their abscissas opposite, and ordinates opposite,

i.e., $\because$ $r' = r$, $x' = -x$, $y' = -y$, always,

$\therefore$ $y' : r' = -y : r$, $x' : r' = -x : r$, $y' : x' = -y : -x$,

i.e., $\sin \overline{\pi+\theta} = -\sin\theta$, $\cos \overline{\pi+\theta} = -\cos\theta$, $\tan \overline{\pi+\theta} = \tan\theta$,

and so for the other functions. Q. E. D.

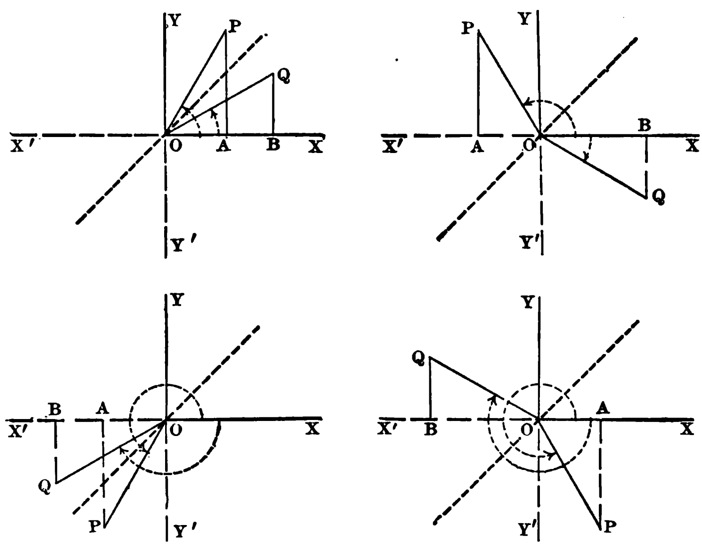
NOTE. Since $\pi + \theta$ is the supplement of $-\theta$, the theorem follows directly from ths. 5, 6.

FUNCTIONS OF THE COMPLEMENT OF AN ANGLE.

Theor. 8. *If θ be any plane angle, then:*

$$\sin \text{co-}\theta = \cos\theta,\quad \cos\text{co-}\theta = \sin\theta,\quad \tan\text{co-}\theta = \cot\theta,$$
$$\csc\text{co-}\theta = \sec\theta,\quad \sec\text{co-}\theta = \csc\theta,\quad \cot\text{co-}\theta = \tan\theta.$$

For, let XOP be any plane angle θ; take Q symmetric with P as to the bisector of the first quarter, so that YOQ $= -\theta$;



then $\because$ XOQ $= \frac{1}{2}\pi - \theta$, and OQ = OP,

and the abscissa of Q is equal to the ordinate of P, and the ordinate of Q to the abscissa of P,

i.e., $\because$ $r' = r$, $x' = y$, $y' = x$, always,

$\therefore$ $y' : r' = x : r$, $x' : r' = y : r$, $y' : x' = x : y$,

i.e., $\sin\text{co-}\theta = \cos\theta$, $\cos\text{co-}\theta = \sin\theta$, $\tan\text{co-}\theta = \cot\theta$,

and so for the other functions. Q. E. D.

Note. The word cosine stands for complement-sine, sine of complement; so for cotangent and cosecant.

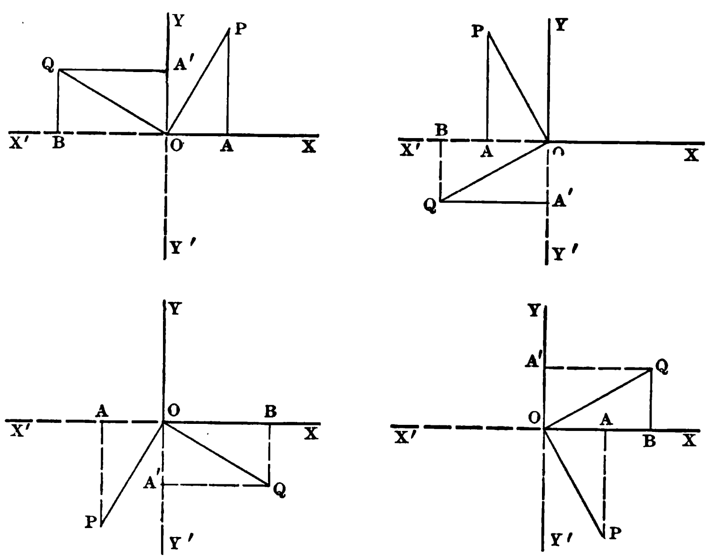
FUNCTIONS OF $\frac{1}{2}\pi + \theta$.

Theor. 9. *If θ be any plane angle, then:*

$$\sin\overline{\tfrac{1}{2}\pi+\theta} = +\cos\theta,\quad \cos\overline{\tfrac{1}{2}\pi+\theta} = -\sin\theta,\quad \tan\overline{\tfrac{1}{2}\pi+\theta} = -\cot\theta,$$

$$\csc\overline{\tfrac{1}{2}\pi+\theta} = +\sec\theta,\quad \sec\overline{\tfrac{1}{2}\pi+\theta} = -\csc\theta,\quad \cot\overline{\tfrac{1}{2}\pi+\theta} = -\tan\theta.$$

For, let XOP be any plane angle θ, draw OQ so that YOQ = XOP, and take OQ = OP;



then $\because$ XOQ $= \frac{1}{2}\pi + \theta$, and OQ = OP,

and the abscissa and ordinate of P as to OX are equal to the abscissa and ordinate of Q as to OY, and equal to the ordinate and the opposite of the abscissa of Q as to OX,

i.e., $\because$ $r' = r$, $x' = -y$, $y' = x$, always,

$\therefore$ $y' : r' = x : r$, $x' : r' = -y : r$, $y' : x' = -x : y$,

i.e., $\sin\overline{\tfrac{1}{2}\pi+\theta} = \cos\theta$, $\cos\overline{\tfrac{1}{2}\pi+\theta} = -\sin\theta$,

and so for the other functions. Q. E. D.

EXAMPLES.

1...6. In functions of positive angles less than a right angle express the values of:

1. $\sin 135°$, $335°$, $-535°$, $-735°$, $\frac{27}{5}\pi$, $-\frac{29}{7}\pi$.
2. $\cos 235°$, $435°$, $-635°$, $-835°$, $\frac{29}{5}\pi$, $-\frac{31}{7}\pi$.
3. $\tan 335°$, $535°$, $-735°$, $-935°$, $\frac{31}{5}\pi$, $-\frac{33}{7}\pi$.
4. $\cot 435°$, $635°$, $-835°$, $-1035°$, $\frac{33}{5}\pi$, -5π.
5. $\sec 535°$, $735°$, $-935°$, $-1135°$, 7π, $-\frac{37}{7}\pi$.
6. $\csc 635°$, $835°$, $-1035°$, $-1235°$, $\frac{37}{5}\pi$, $-\frac{39}{7}\pi$.

7...12. In functions of zero or of positive angles not greater than half a right angle express the values of:

7. $\sin\ 50°$, $150°$, $-250°$, $-350°$, $\frac{3}{12}\pi$, -4π.
8. $\cos\ 60°$, $160°$, $-260°$, $-360°$, $\frac{5}{12}\pi$, $-\frac{14}{3}\pi$.
9. $\tan\ 70°$, $170°$, $-270°$, $-370°$, $\frac{7}{12}\pi$, $-\frac{16}{3}\pi$.
10. $\cot\ 80°$, $180°$, $-280°$, $-380°$, $\frac{9}{12}\pi$, -6π.
11. $\sec\ 90°$, $190°$, $-290°$, $-390°$, $\frac{11}{12}\pi$, $-\frac{20}{3}\pi$.
12. $\csc 100°$, $200°$, $-300°$, $-400°$, $\frac{13}{12}\pi$, $-\frac{22}{3}\pi$.

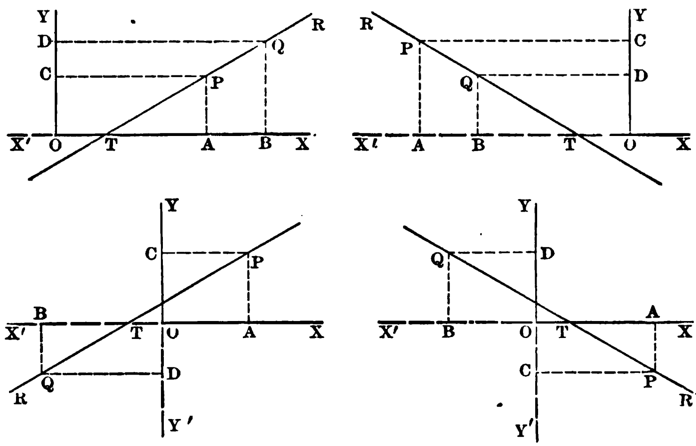
13...16. Find all the values of θ:

13. when $\sin\theta = -\sin\alpha$, $\sqrt{\sin^2\alpha}$, $\sqrt{\cos^2\alpha}$.
14. when $\cos\theta = -\cos\alpha$, $\sqrt{\cos^2\alpha}$, $\sqrt{\sin^2\alpha}$.
15. when $\tan\theta = -\tan\alpha$, $\sqrt{\tan^2\alpha}$, $\sqrt{\cot^2\alpha}$.
16. when $\cot\theta = -\cot\alpha$, $\sqrt{\cot^2\alpha}$, $\sqrt{\tan^2\alpha}$.
17. A function of the sum or difference of θ and an $\left\{\begin{matrix}\text{even}\\ \text{odd}\end{matrix}\right.$ multiple of $\frac{1}{2}\pi$ is as large as the $\left\{\begin{matrix}\text{same}\\ \text{co-}\end{matrix}\right.$ function of θ.
18. Prove $\text{vers}\,(180 - \text{A}) + \text{vers}\,(360 - \text{A}) = 2$.
19. Prove $\cos^2\text{A} + \cos^2(90 + \text{A}) + \cos^2(180 + \text{A}) + \cos^2(270 + \text{A}) = 2$.
20. What values of x satisfy the equation $\sin 2x = \cos 3x$?

§ 5. PROJECTIONS.

THEOR. 10. *The* $\left\{\begin{matrix}x\text{-}projection\\ y\text{-}projection\end{matrix}\right.$ *of a limited straight line is the product of the line by the* $\left\{\begin{matrix}cosine\\ sine\end{matrix}\right.$ *of its angle with the x-axis.*

For, let OX, OY be the axes, let TR be any line meeting OX in T
and let PQ be any segment of TR;
draw AP, BQ perpendicular to OX, and CP, DQ perpendicular to OY;



then $\quad x$-proj. PQ = AB = TB − TA

= (TQ − TP) cos XTR = PQ cos XTR, [df. cos.

and $\quad y$-proj. PQ = CD = OD − OC

= (TQ − TP) sin XTR = PQ sin XTR. Q. E. D.

COR. *The* $\left\{\begin{matrix}x\text{-}projection\\ y\text{-}projection\end{matrix}\right.$ *of a broken line is equal to the sum of the products of the parts, each multiplied by the* $\left\{\begin{matrix}cosine\\ sine\end{matrix}\right.$ *of its angle with the x-axis.*

EXAMPLE. Let l, m be two lines, and a a segment of l; project a on m, this projection on l, this on m, and so on, and find the sum of all the projections on m.

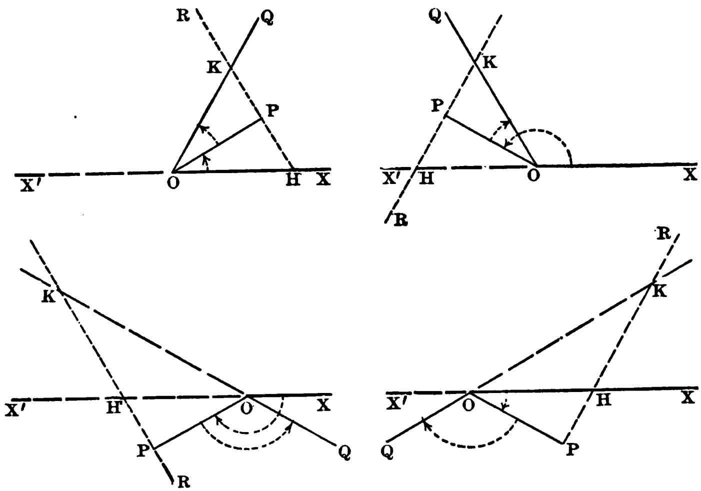
§ 6. FUNCTIONS OF SUMS AND OF DIFFERENCES.

THEOR. 11. (Addition-theorem.) *If θ, θ' be any two plane angles,*

then $\sin \overline{\theta \pm \theta'} = \sin\theta \cos\theta' \pm \cos\theta \sin\theta'$,

$\cos \overline{\theta \pm \theta'} = \cos\theta \cos\theta' \mp \sin\theta \sin\theta'$,

For, let XOP, POQ be any two plane angles θ, θ';
through P draw PR normal to OP, cutting OX, OQ in H, K;



then $\because$ XOQ $= \theta + \theta'$, XHR $= \frac{1}{2}\pi + \theta$, [constr.

and OK $\sin \overline{\theta + \theta'} =$ OP $\sin\theta +$ PK $\sin \overline{\tfrac{1}{2}\pi + \theta}$ [th. 10, cr.

$=$ OP $\sin\theta +$ PK $\cos\theta$, [th. 9

and OP $=$ OK $\cos\theta'$, PK $=$ OK $\sin\theta'$, [df.

$\therefore$ OK $\sin\overline{\theta + \theta'} =$ OK $\sin\theta\cos\theta' +$ OK $\cos\theta\sin\theta'$,

$\therefore$ $\sin\overline{\theta + \theta'} = \sin\theta\cos\theta' + \cos\theta\sin\theta'$; Q. E. D.

$\therefore$ $\sin\overline{\theta - \theta'} = \sin\theta\cos\overline{-\theta'} + \cos\theta\sin\overline{-\theta'}$

$= \sin\theta\cos\theta' - \cos\theta\sin\theta'$. Q. E. D. [th. 5

So, $\because$ OK $\cos\overline{\theta + \theta'} =$ OP $\cos\theta +$ PK $\cos\overline{\tfrac{1}{2}\pi + \theta}$

$=$ OP $\cos\theta -$ PK $\sin\theta$

$=$ OK $\cos\theta\cos\theta' -$ OK $\sin\theta\sin\theta'$,

$\therefore\ \cos\overline{\theta+\theta'} = \cos\theta\cos\theta' - \sin\theta\sin\theta'$;

$\therefore\ \cos\overline{\theta-\theta'} = \cos\theta\cos\theta' + \sin\theta\sin\theta'$. [as above

COR. 1.
$$\begin{aligned} \sin(\theta+\theta') + \sin(\theta-\theta') &= \ \ 2\sin\theta\cos\theta', \\ \sin(\theta+\theta') - \sin(\theta-\theta') &= \ \ 2\cos\theta\sin\theta', \\ \cos(\theta+\theta') + \cos(\theta-\theta') &= \ \ 2\cos\theta\cos\theta', \\ \cos(\theta+\theta') - \cos(\theta-\theta') &= -2\sin\theta\sin\theta'. \end{aligned}$$

COR. 2. $\tan\overline{\theta\pm\theta'} = \dfrac{\tan\theta \pm \tan\theta'}{1 \mp \tan\theta\tan\theta'}$.

For $\tan\overline{\theta\pm\theta'} = \dfrac{\sin\theta\cos\theta' \pm \cos\theta\sin\theta'}{\cos\theta\cos\theta' \mp \sin\theta\sin\theta'}$ [th. 4, above

$= \dfrac{\tan\theta \pm \tan\theta'}{1 \mp \tan\theta\tan\theta'}$. [div. by $\cos\theta\cos\theta'$

The reader will note the complete generality of ths. 10, 11, and the consequent generality of all the theorems and corollaries that follow and depend upon them. Considering the great importance of th. 11, he should write out each of the four equations involved, so that it may stand out clearly by itself; and he should translate these equations into words and commit them to memory. This suggestion as to translation may apply to many other theorems that are given only in formula.

THEOR. 12. *If θ, θ' be any two plane angles, then:*
$$\begin{aligned} \sin\theta + \sin\theta' &= \ \ 2\sin\tfrac{1}{2}\,\overline{\theta+\theta'}\cos\tfrac{1}{2}\,\overline{\theta-\theta'}, \\ \sin\theta - \sin\theta' &= \ \ 2\cos\tfrac{1}{2}\,\overline{\theta+\theta'}\sin\tfrac{1}{2}\,\overline{\theta-\theta'}, \\ \cos\theta + \cos\theta' &= \ \ 2\cos\tfrac{1}{2}\,\overline{\theta+\theta'}\cos\tfrac{1}{2}\,\overline{\theta-\theta'}, \\ \cos\theta - \cos\theta' &= -2\sin\tfrac{1}{2}\,\overline{\theta+\theta'}\sin\tfrac{1}{2}\,\overline{\theta-\theta'}. \end{aligned}$$

For $\because\ \tfrac{1}{2}\overline{\theta+\theta'} + \tfrac{1}{2}\overline{\theta-\theta'} = \theta,\quad \tfrac{1}{2}\overline{\theta+\theta'} - \tfrac{1}{2}\overline{\theta-\theta'} = \theta'$,

and $\sin(\tfrac{1}{2}\overline{\theta+\theta'} + \tfrac{1}{2}\overline{\theta-\theta'}) + \sin(\tfrac{1}{2}\overline{\theta+\theta'} - \tfrac{1}{2}\overline{\theta-\theta'})$

$= 2\sin\tfrac{1}{2}\overline{\theta+\theta'}\cos\tfrac{1}{2}\overline{\theta-\theta'}$, [th. 11 cr. 1

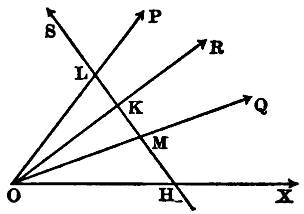
$\therefore\ \sin\theta + \sin\theta' = 2\sin\tfrac{1}{2}\overline{\theta+\theta'}\cos\tfrac{1}{2}\overline{\theta-\theta'}$;

and so for the rest. Q. E. D.

These formulæ serve to convert a sum into a product, and they are called the *conversion-formulæ*.

GEOMETRIC PROOF. Let XOP, XOQ be any angles θ, θ', and let OR bisect $\angle$ QOP;

draw KS normal to OR, and cutting OX, OP, OQ, OR in H, L, M, K;



then $\because$ $\angle \text{XOR} = \frac{1}{2}(\theta + \theta')$, $\angle \text{ROP} = \frac{1}{2}(\theta - \theta')$,

$\angle \text{RKS} = \frac{1}{2}\pi$, $\angle \text{XHL} = \frac{1}{2}\pi + \frac{1}{2}(\theta + \theta')$,

and x-proj. OL $-$ x-proj. OM $= 2x$-proj. KL,

$\therefore$ $\text{OL} \cos\theta - \text{OM} \cos\theta' = 2\,\text{KL} \cos\left[\frac{1}{2}\pi + \frac{1}{2}(\theta + \theta')\right]$

$= -2\,\text{OL} \sin\frac{1}{2}(\theta - \theta') \sin\frac{1}{2}(\theta + \theta')$;

and $\because$ $\text{OL} = \text{OM}$,

$\therefore$ $\cos\theta - \cos\theta' = -2\sin\frac{1}{2}(\theta + \theta')\sin\frac{1}{2}(\theta - \theta')$;

and so for the other three equations. Q. E. D.

EXAMPLES.

1. If θ, θ' be any two plane angles, show that:

$$\sin\overline{\theta + \theta'}\sin\overline{\theta - \theta'} = \sin^2\theta - \sin^2\theta' = \cos^2\theta' - \cos^2\theta,$$
$$\cos\overline{\theta + \theta'}\cos\overline{\theta - \theta'} = \cos^2\theta - \sin^2\theta' = \cos^2\theta' - \sin^2\theta.$$

2. Divide $\sin\overline{\theta \pm \theta'}$, $\cos\overline{\theta \pm \theta'}$ each in turn by:

$$\cos\theta\cos\theta',\quad \sin\theta\sin\theta',\quad \cos\theta\sin\theta',\quad \sin\theta\cos\theta';$$

and express the results in terms of $\tan\theta$, $\tan\theta'$, $\cot\theta$, $\cot\theta'$.

3. In terms of tangents and cotangents, find the values of:

$$\frac{\sin\theta + \sin\theta'}{\cos\theta + \cos\theta'},\quad \frac{\sin\theta - \sin\theta'}{\cos\theta + \cos\theta'},\quad \frac{\sin\theta + \sin\theta'}{\cos\theta - \cos\theta'},$$
$$\frac{\sin\theta - \sin\theta'}{\cos\theta - \cos\theta'},\quad \frac{\sin\theta + \sin\theta'}{\sin\theta - \sin\theta'},\quad \frac{\cos\theta + \cos\theta'}{\cos\theta - \cos\theta'}.$$

4. If $\sin\theta = .2$, $\sin\theta' = .3$; find $\sin(\theta \pm \theta')$, $\cos(\theta \pm \theta')$.

5. From the functions of 30° and 45° find those of 15°, 75°, 105°, 165°, 195°, 255°, 285°, 345°.

6. $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1} [x\sqrt{(1-y^2)} \pm y\sqrt{(1-x^2)}]$.

7. $\cos^{-1} x \pm \cos^{-1} y = \cos^{-1} [xy \mp \sqrt{(1-x^2-y^2+x^2y^2)}]$.

8. $\tan^{-1} x \pm \tan^{-1} y = \tan^{-1} [(x \pm y) : (1 \mp xy)]$.

9. $\frac{1}{2}\pi = \sin^{-1}\frac{3}{5} + \sin^{-1}\frac{4}{5} = \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{12}{13}$.

10. $\frac{1}{4}\pi = \tan^{-1} 1 = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$.

$= \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8} = 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}$.

11. Show that
$$\begin{aligned}
\sin 28° + \sin 14° &= 2 \sin 21° \cos 7°,\\
\sin 28° - \sin 14° &= 2 \cos 21° \sin 7°,\\
\cos 28° + \cos 14° &= 2 \cos 21° \cos 7°,\\
\cos 28° - \cos 14° &= -2 \sin 21° \sin 7°,\\
\sin 80° - \sin 20° &= \cos 50°,\\
\sin 75° - \sin 45° &= \sin 15°.
\end{aligned}$$

12. In ex. 3, let $\theta = 60°$, $\theta' = 45°$; find $\tan 52° 30'$, $\tan 7° 30'$.

13. Prove the identities:

$\sin\theta + \sin 3\theta + \sin 5\theta + \sin 7\theta = 4 \sin 4\theta \cos 2\theta \cos\theta$,

$\cos\theta + \cos(120 + \theta) + \cos(120 - \theta) = 0$,

$\sin^2 10° - \cos^2 190° = \cos 200°$.

14. Solve the equations: $\cos 3\theta + \cos 2\theta + \cos\theta = 0$,

$\sin^{-1} 3x + \sin^{-1} 4x = \frac{1}{2}\pi$, $\tan^{-1} 2x + \tan^{-1} 3x = \frac{1}{4}\pi$.

15. If A, B, C, D be any four plane angles, $\sin(A-B)\sin(C-D) + \sin(B-C)\sin(A-D) + \sin(C-A)\sin(B-D) = 0$.

§ 7. FUNCTIONS OF DOUBLE ANGLES AND HALF-ANGLES.

THEOR. 13. *If θ be any plane angle, then:*

$$\sin 2\theta = 2 \sin\theta \cos\theta,$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta,$$

$$\tan 2\theta = 2\tan\theta : (1 - \tan^2\theta).$$

For $\sin 2\theta = \sin \overline{\theta + \theta}$

$= \sin\theta \cos\theta + \cos\theta \sin\theta$ [th. 11

$= 2 \sin\theta \cos\theta.$ Q. E. D.

So $\cos 2\theta = \cos \overline{\theta + \theta}$

$= \cos\theta \cos\theta - \sin\theta \sin\theta$ [th. 11

$= \cos^2\theta - \sin^2\theta$

$= \cos^2\theta - (1 - \cos^2\theta) = 2\cos^2\theta - 1,$

$= (1 - \sin^2\theta) - \sin^2\theta = 1 - 2\sin^2\theta.$ Q. E. D.

So $\tan 2\theta = \tan \overline{\theta + \theta}$

$= \dfrac{\tan\theta + \tan\theta}{1 - \tan\theta \tan\theta} = \dfrac{2\tan\theta}{1 - \tan^2\theta}.$ Q.E.D. [th. 11 cr. 2

Cor. *$\sin\theta = 2\sin\frac{1}{2}\theta \cos\frac{1}{2}\theta,$*

$\cos\theta = 2\cos^2\frac{1}{2}\theta - 1 = 1 - 2\sin^2\frac{1}{2}\theta;$

$1 + \cos\theta = 2\cos^2\frac{1}{2}\theta,\quad 1 - \cos\theta = 2\sin^2\frac{1}{2}\theta.$

Theor. 14. *If θ be any plane angle, then:*

$$\sin\tfrac{1}{2}\theta = \sqrt{\frac{1-\cos\theta}{2}},\quad \cos\tfrac{1}{2}\theta = \sqrt{\frac{1+\cos\theta}{2}},$$

$$\tan\tfrac{1}{2}\theta = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}.$$

For $\because 2\sin^2\frac{1}{2}\theta = 1 - \cos\theta,$ [th. 13 cr.

$\therefore \sin\frac{1}{2}\theta = \sqrt{\dfrac{1-\cos\theta}{2}}.$ Q. E. D.

So $\because 2\cos^2\frac{1}{2}\theta = 1 + \cos\theta,$

$\therefore \cos\frac{1}{2}\theta = \sqrt{\dfrac{1+\cos\theta}{2}}.$ Q. E. D.

And $\tan\frac{1}{2}\theta = \dfrac{\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta} = \sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}},$

$= \dfrac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\cos^2\frac{1}{2}\theta} = \dfrac{\sin\theta}{1+\cos\theta},$ [th. 13 cr.

$= \dfrac{2\sin^2\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta} = \dfrac{1-\cos\theta}{\sin\theta}.$ Q. E. D.

EXAMPLES.

1. $\sin(2\sin^{-1}x) = 2x\sqrt{(1-x^2)}$.
2. $\tan(2\tan^{-1}x) = 2x : (1-x^2)$.

3...5. If $A + B + C = \pi$, prove that:

3. $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$.
4. $\sin A + \sin B + \sin C = 4 \cos\frac{1}{2}A \cos\frac{1}{2}B \cos\frac{1}{2}C$.
5. $\cos A + \cos B + \cos C = 4 \sin\frac{1}{2}A \sin\frac{1}{2}B \sin\frac{1}{2}C + 1$.
6. $\sin\theta = 2\tan\frac{1}{2}\theta : (1+\tan^2\frac{1}{2}\theta)$;
   $\cos\theta = (1-\tan^2\frac{1}{2}\theta) : (1+\tan^2\frac{1}{2}\theta)$.
7. If $a\sin\theta + b\cos\theta = c$, find $\tan\frac{1}{2}\theta$.
8. $\sin\frac{1}{8}\pi = \frac{1}{2}\sqrt{(2-\sqrt{2})}$, $\cos\frac{1}{8}\pi = \frac{1}{2}\sqrt{(2+\sqrt{2})}$.
9. If $\tan A = \sqrt{3}$, find the two values of $\tan\frac{1}{2}A$.

10...13. Prove the identities:

10. $\csc 2\theta + \cot 2\theta = \cot\theta$; $\cos\theta = \cos^4\frac{1}{2}\theta - \sin^4\frac{1}{2}\theta$.
11. $\tan(\frac{1}{4}\pi - \frac{1}{2}\theta) + \cot(\frac{1}{4}\pi - \frac{1}{2}\theta) = 2\sec\theta$.
12. $(\cos\theta + \sin\theta) : (\cos\theta - \sin\theta) = \tan 2\theta + \sec 2\theta$.
13. $\tan^2(\frac{1}{4}\pi + \frac{1}{2}\theta) = (\sec\theta + \tan\theta) : (\sec\theta - \tan\theta)$.
14. If $(1+\epsilon\cos\theta)(1-\epsilon\cos\phi) = 1-\epsilon^2$, show that
    $\tan\frac{1}{2}\theta : \tan\frac{1}{2}\phi = \sqrt{(1+\epsilon)} : \sqrt{(1-\epsilon)}$.

*§ 8. FUNCTIONS OF MULTIPLE ANGLES.

☞ For a full discussion of imaginaries, see O. W. J. alg., X.

THEOR. 15. *If $\theta_1, \theta_2, \cdots \theta_n$ be any n plane angles, and if i stand for $\sqrt{-1}$, then*

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\cdots(\cos\theta_n + i\sin\theta_n)$$
$$= \cos(\theta_1+\theta_2+\cdots+\theta_n) + i\sin(\theta_1+\theta_2+\cdots+\theta_n).$$

1. *The theorem is true when $n = 2$.*

For $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$$=(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)$$
$$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2). \qquad \text{Q.E.D.} \quad [\text{th. } 11$$

2. *If the theorem be true for $n = k$, it is true for $n = k + 1$.*

For if $(\cos\theta_1 + i\sin\theta_1)\cdots(\cos\theta_k + i\sin\theta_k)$

$$= \cos(\theta_1 + \cdots + \theta_k) + i\sin(\theta_1 + \cdots + \theta_k), \qquad [\text{hyp.}$$

then $(\cos\theta_1 + i\sin\theta_1)\cdots(\cos\theta_k + i\sin\theta_k)(\cos\theta_{k+1} + i\sin\theta_{k+1})$

$$= [\cos(\theta_1 + \cdots + \theta_k) + i\sin(\theta_1 + \cdots + \theta_k)](\cos\theta_{k+1} + i\sin\theta_{k+1})$$

$$= \cos(\theta_1 + \cdots + \theta_k + \theta_{k+1}) + i\sin(\theta_1 + \cdots + \theta_k + \theta_{k+1}).$$

Q. E. D.

3. *The theorem is true universally.*

For $\because$ it is true for $n = 2$, [1

$\therefore$ it is true for $n = 2+1 = 3$, for $3+1 = 4$, $\cdots$. Q. E. D. [2

Cor. $\sin(\theta_1 + \theta_2 + \cdots + \theta_n)$

= *coeff. of* i *in* $(\cos\theta_1 + i\sin\theta_1)\cdots(\cos\theta_n + i\sin\theta_n)$,

and $\cos(\theta_1 + \theta_2 + \cdots + \theta_n)$

= *real part of* $(\cos\theta_1 + i\sin\theta_1)\cdots(\cos\theta_n + i\sin\theta_n)$.

E.g., $\sin(\theta_1 + \theta_2 + \theta_3) = \sin\theta_1\cos\theta_2\cos\theta_3 + \cos\theta_1\sin\theta_2\cos\theta_3 + \cos\theta_1\cos\theta_2\sin\theta_3 - \sin\theta_1\sin\theta_2\sin\theta_3$,

and $\cos(\theta_1 + \theta_2 + \theta_3) = \cos\theta_1\cos\theta_2\cos\theta_3 - \cos\theta_1\sin\theta_2\sin\theta_3 - \sin\theta_1\cos\theta_2\sin\theta_3 - \sin\theta_1\sin\theta_2\cos\theta_3$.

Theor. 16. (De Moivre's theorem.) *If n be any commensurable number, then:*

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

(*a*) *n a positive integer:*

A corollary of th. 15, with the angles all equal.

(*b*) *n a negative integer, $-m$:*

For $\because$ $(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = \cos^2\theta + \sin^2\theta = 1$,

and $(\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)^{-1} = 1$, [df. neg. pwr.

$\therefore$ $(\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta = \cos\overline{-\theta} + i\sin\overline{-\theta}$.

$\therefore$ $(\cos\theta + i\sin\theta)^n = [(\cos\theta + i\sin\theta)^{-1}]^m$

$$= (\cos\overline{-\theta} + i\sin\overline{-\theta})^m$$

$$= \cos\overline{-m\theta} + i\sin\overline{-m\theta} \qquad [(a)$$

$$= \cos n\theta + i\sin n\theta. \qquad \text{Q. E. D.}$$

(c) *n a fraction, $\frac{p}{q}$, positive or negative:*

For $\because\ \cos\theta + i\sin\theta = \left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^q$, [(a)

$\therefore$ one value of $(\cos\theta + i\sin\theta)^{\frac{1}{q}}$ is $\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}$,

$\therefore$ one value of $(\cos\theta + i\sin\theta)^{\frac{p}{q}}$ is $\cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q}$.

Q. E. D.

COR. $\sin n\theta =$ *coeff. of i in* $(\cos\theta + i\sin\theta)^n$

$$= n\sin\theta\cos^{n-1}\theta - C_3 n\sin^3\theta\cos^{n-3}\theta + C_5 n\sin^5\theta\cos^{n-5}\theta - \cdots,$$

$\cos n\theta =$ *real part of* $(\cos\theta + i\sin\theta)^n$

$$= \cos^n\theta - C_2 n\sin^2\theta\cos^{n-2}\theta + C_4 n\sin^4\theta\cos^{n-4}\theta - \cdots;$$

wherein $C_r n \equiv n(n-1)\cdots(n-r+1) : r!$, *the number of combinations of n things taken r at a time.*

E.g.,

$$\sin 3\theta = 3\sin\theta\cos^2\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta,$$

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = -3\cos\theta + 4\cos^3\theta,$$

$$\sin 4\theta = 4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta = 4\sin\theta\cos\theta - 8\sin^3\theta\cos\theta,$$

$$\cos 4\theta = \cos^4\theta - 6\sin^2\theta\cos^2\theta + \sin^4\theta = 1 - 8\cos^2\theta + 8\cos^4\theta,$$

$$\sin 5\theta = 5\sin\theta\cos^4\theta - 10\sin^3\theta\cos^2\theta + \sin^5\theta = 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta,$$

$$\cos 5\theta = \cos^5\theta - 10\sin^2\theta\cos^3\theta + 5\sin^4\theta\cos\theta = 5\cos\theta - 20\cos^3\theta + 16\cos^5\theta.$$

EXAMPLES.

1...3. If $A + B + C = \pi$, prove that:

1. $\tan\frac{1}{2}A\tan\frac{1}{2}B + \tan\frac{1}{2}B\tan\frac{1}{2}C + \tan\frac{1}{2}C\tan\frac{1}{2}A = 1$.
2. $\cot A + \cot B + \cot C = \cot A\cot B\cot C + \csc A\csc B\csc C$.
3. $\tan A + \tan B + \tan C = \tan A\tan B\tan C$.
4. $\sin(3\sin^{-1}x) = 3x - 4x^3$. $\cos(3\cos^{-1}x) = -3x + 4x^3$.
5. $\tan(3\tan^{-1}x) = (3x - x^3) : (1 - 3x^2)$.

6. $\tan 3\theta = (\sin\theta + \sin 3\theta + \sin 5\theta) : (\cos\theta + \cos 3\theta + \cos 5\theta)$.

7. If θ, θ' be any two plane angles, and n any integer, then

$$[\sin\theta + \sin(\theta+\theta') + \sin(\theta+2\theta') + \sin(\theta+3\theta') + \cdots + \sin(\theta+\overline{n-1}\,\theta')]\cdot 2\sin\tfrac{1}{2}\theta' = \cos(\theta-\tfrac{1}{2}\theta') - \cos(\theta+\overline{n-\tfrac{1}{2}}\,\theta'),$$

$$[\cos\theta + \cos(\theta+\theta') + \cos(\theta+2\theta') + \cos(\theta+3\theta') + \cdots + \cos(\theta+\overline{n-1}\,\theta')]\cdot 2\sin\tfrac{1}{2}\theta' = \sin(\theta+\overline{n-\tfrac{1}{2}}\,\theta') - \sin(\theta-\tfrac{1}{2}\theta').$$

8. From the results of ex. 7, prove that:

$$\sin\theta + \sin\left(\theta+\frac{2}{n}\pi\right) + \cdots + \sin\left(\theta+\frac{2(n-1)}{n}\pi\right) = 0.$$

$$\cos\theta + \cos\left(\theta+\frac{2}{n}\pi\right) + \cdots + \cos\left(\theta+\frac{2(n-1)}{n}\pi\right) = 0,$$

wherein n is any positive integer.

9. In the results of ex. 8, take $n = 3$, and prove that:

$$\sin\theta + \sin\overline{60^\circ-\theta} - \sin\overline{60^\circ+\theta} = 0,$$
$$\cos\theta - \cos\overline{60^\circ-\theta} - \cos\overline{60^\circ+\theta} = 0.$$

10. In the results of ex. 8, take $n = 5$, and prove that:

$$\sin\theta + \sin\overline{72^\circ+\theta} + \sin\overline{36^\circ-\theta} - \sin\overline{36^\circ+\theta} - \sin\overline{72^\circ-\theta} = 0,$$
$$\cos\theta + \cos\overline{72^\circ+\theta} - \cos\overline{36^\circ-\theta} - \cos\overline{36^\circ+\theta} + \cos\overline{72^\circ-\theta} = 0.$$

11. Show that the formula found when $n = 3$ verifies the sines and cosines of all angles in the first quarter, if to θ be given values from 0° to 30°; and when $n = 5$, if to θ be given values from 0° to 18°.

12. In the results of ex. 8, take $n = 9$, 15, 25, 27, 45, and thence find other formulæ of verification.

13. Prove that $\tan(\theta_1 + \theta_2 + \theta_3 + \cdots)$

$$= (s_1 - s_3 + s_5 - \cdots) : (1 - s_2 + s_4 - \cdots),$$

wherein $s_r \equiv$ the sum of the products of $\tan\theta_1$, $\tan\theta_2$, $\tan\theta_3$, $\cdots$, taken r at a time.

14. If $\theta_1 + \theta_2 + \theta_3 + \theta_4 = m\pi$, m any integer, then $s_1 = s_3$.

III. PLANE TRIANGLES.

§ 1. TRIGONOMETRIC TABLES.

The values of the trigonometric functions of certain angles have been given [II § 1]; and by methods shown later [V § 3] they may be computed for any angle. Such values taken at regular intervals, and arranged in order for convenient use, form a *trigonometric table*. The functions themselves, by way of distinction, are called *natural functions*, and their logarithms, *logarithmic functions*.

In making up and using the tables the following principles are of service:

1. The functions of an angle and the co-functions of the complementary angle are identical. [II th. 8

2. The functions of an angle and of its supplement are either equal or opposite. [II th. 6

3. The functions of an angle are equal or opposite to the co-functions of the same angle less a right angle.

4. The functions of every angle are equal or opposite to the functions of some angle in the first quarter. [II ths. 2, 5–9

5. For angles that differ very little, and are not near the end of a quarter, the differences of any function are very nearly proportional to the differences of the angles. The reader may observe the property last named if he will examine the tables.

The tables are therefore made up for angles in the first quarter only, and they are so arranged that every tabular number stands for the co-functions of a pair of complementary angles, the one lying between 0° and 45°, the the other between 45° and 90°.

E.g., $.44098 \equiv$ nat-sin $26° 10' \equiv$ nat-cos $63° 50'$.

So, $.24415 \equiv$ log-tan $60° 20' \equiv$ log-cot $29° 40'$.

In an ordinary table, giving the functions at intervals of single minutes, there are several columns of functions: one with the word *sine* at the top and *cosine* at the bottom, another with the word *cosine* at the top and *sine* at the bottom, and so on.

Of the pairs of complementary angles in the first quarter that correspond to any one value, one, generally that which is less than 45°, has its degrees at the top and minutes at the left side of the page; the other, greater than 45°, has its degrees at the bottom and minutes at the right side of the page.

The names of functions standing at the $\left\{\begin{matrix}\text{top}\\\text{bottom}\end{matrix}\right.$ of a column go with the degrees at the $\left\{\begin{matrix}\text{top.}\\\text{bottom.}\end{matrix}\right.$ If angles of both the first and second quarters appear in the table the minutes at the $\left\{\begin{matrix}\text{left}\\\text{right}\end{matrix}\right.$ side of the page go with the degrees at the $\left\{\begin{matrix}\text{left}\\\text{right}\end{matrix}\right.$ side.

In some tables columns of *differences* for seconds follow the columns of functions; they are found by dividing the differences of consecutive logarithms, *i.e.*, the differences for minutes, by 60. Their use appears in an example on page 45. Logarithms with negative characteristics are usually modified by adding 10.

PROB. 1. TO TAKE OUT A FUNCTION OF A GIVEN ANGLE.

Find an angle in the first quarter whose functions, or co-functions, are the same (numerically) *as the functions of the given angle, then;*

(*a*) *If the angle be given in degrees and minutes:*

Find the degrees at the $\left\{\begin{matrix}\textit{top}\\\textit{bottom}\end{matrix}\right.$ *of the page and the minutes at the* $\left\{\begin{matrix}\textit{left}\\\textit{right}\end{matrix}\right.$ *side; and* $\left\{\begin{matrix}\textit{under}\\\textit{over}\end{matrix}\right.$ *the name of the function and opposite the minutes read the function sought.*

E.g., to take out nat-sin 26° 10′:

Under *nat-sin* and 26° and opposite 10′ on the left, read .44098.

So, to take out nat-cos 116° 10′:

Find sup. 116° 10′, *i.e.*, 63° 50′; then over *nat-cosine* and 63° and opposite 50′ on the right, read $^-.44098$;

or, better, throw out 90°, and take out nat-sin 26° 10′.

So, to take out log-tan 60° 20′:

Over *log tangent* and 60° and opposite 20 on the right, read 0.244415.

So, to take out log-cot 150° 20′:

Find sup. 150° 20′, *i.e.*, 29° 40′; then under *log-cotangent* and 29° and opposite 40′ on the left, read 0.244415 neg.;

wherein the word "neg." shows that the natural cotangent is negative and that 0.244415 is the logarithm of its numerical value;

or, better, take out log [− tan (150° 20′ − 90°)].

(*b*) *If the angle be given in degrees, minutes, and seconds:*

Take out the functions of the two nearest tabular angles between which the given angle lies, as in (*a*)*; and to the function of the less angle add such part of the excess* (positive or negative) *of the function of the greater angle over that of the less as the seconds are a part of one minute.*

In sines and tangents this excess is positive and the *correction* is really added; in cosines and cotangents it is negative and the correction is subtracted.

E.g., to take out nat-sin 26° 10′ 25″:

then ∵ nat-sin 26° 10′ = .44098, nat-sin 26° 11′ = .44124,

and .44124 − .44098 = .00026,

$$\therefore \text{nat-sin } 26° \, 10' \, 25'' = .44098 + 26 \cdot 25 : 60$$
$$= .44098 + .00011 = .44109.$$

In practice write only the function of the less angle, make the subtraction mentally, and apply the proportional correction:

$$\begin{array}{r} .44098 \\ +\quad 11 \\ \hline .44109 \end{array}$$

So, to take out nat-cos 63° 49′ 35″:

$$\begin{array}{r} .44124 \\ -\quad 15 \\ \hline .44109 \end{array}$$

So, to take out log-tan $60° 20' 35''$:

$$4.89 \times 35 = 171 \qquad \begin{array}{r} 0.244415 \\ + \quad 171 \\ \hline 0.244586 \end{array}$$

wherein 4.89 is taken from the column of differences of tangents, and is the difference for $1''$ in that part of the table.

So, to take out log-cot $29° 39' 25''$:

$$4.89 \times 25 = 122 \qquad \begin{array}{r} 0.244709 \\ - \quad 122 \\ \hline 0.244587 \end{array}$$

PROB. 2. TO TAKE OUT THE ANGLES CORRESPONDING TO A GIVEN FUNCTION.

(*a*) *The function found exactly in the table:*

If the name of the function be found at the $\left\{\begin{matrix}\textit{top}\\ \textit{bottom}\end{matrix}\right.$ *of the column, read the degrees at the* $\left\{\begin{matrix}\textit{top}\\ \textit{bottom}\end{matrix}\right.$ *of the page and the minutes that lie opposite the function at the* $\left\{\begin{matrix}\textit{left}\\ \textit{right}\end{matrix}\right.$ *side.*

E.g., to take out nat-$\sin^{-1}$ .44098:

then $\because$ .44098 stands under nat-*sine* and 26° and opposite 10′ at the left,

$\therefore$ the angle sought is 26° 10′, or its supplement, 153° 50′, or any angle congruent to either of them.

So, to take out nat-$\cos^{-1}$ $^{-}$.44098:

then $\because$ .44098 stands over nat-*cosine* and 63° and opposite 50′ at the right,

$\therefore$ the angle sought is sup. 63° 50′, *i.e.*, 116° 10′, and its opposite, or any angle congruent to either of them.

So, to take out log-$\tan^{-1}$ 0.244415:

then the angle sought is 60° 20′, or its negative supplement, or any angle congruent to either of them.

So, to take out log-$\cot^{-1}$ 0.244415, neg.:

then the angle sought is sup. 29° 40′, *i.e.*, 150° 20′, or the negative supplement of 150° 20′, or any angle congruent to either of them.

(*b*) *The function not found exactly in the table:*

Take out the next less and the next greater tabular function, and to the smaller angle add such part of sixty seconds as the difference between the function of the less angle and the given function is a part of the difference between the two tabular functions.

E.g., to take out nat-sin^{-1} .44109:

then $\because$ nat-sin^{-1} .44098 $= 26°\ 10'$, nat-sin^{-1} .44124 $= 26°\ 11'$,

$\therefore$ nat-sin^{-1} .44109 $= 26°\ 10' + 60'' \cdot 11 : 26$

$= 26°\ 10'\ 25''$ or $153°\ 49'\ 35''$

So, to take out nat-cos^{-1} $^{-}$.44109:

then $\because$ nat-cos^{-1} .44098 $= 63°\ 50'$, nat-cos^{-1} .44124 $= 63°\ 49'$,

$\therefore$ nat-cos^{-1} .44109 $= 63°\ 49' + 60'' \cdot 15 : 26 = 63°\ 49'\ 35''$,

$\therefore$ nat-cos^{-1} $^{-}$.44109 $=$ sup. $63°\ 49'\ 35'' = 116°\ 10'\ 25''$.

So, to take out log-tan^{-1} 0.244586:

then the angle sought is $60°\ 20' + (171 : 4.89)''$, $= 60°\ 20'\ 35''$.

So, to take out log-cot^{-1} 0.244586, neg.:

then the tabular angle is $29°\ 39' + (122 : 4.89)''$, $= 29°\ 39'\ 25''$,

and the angle sought is sup. $29°\ 39'\ 25''$, *i.e.*, $150°\ 20'\ 35''$.

ANGLES NEAR THE ENDS OF A QUARTER.

Of angles near 90° the sines, both natural and logarithmic, change very slowly; and such angles cannot be found from their sines with exactness. So of the cosines of angles near 0°.

Of angles near 0° the logarithmic sines change very fast and at rapidly varying rates; and such angles cannot be found exactly from their logarithmic sines, nor the logarithmic sines from the angles. So of the logarithmic cosines of angles near 90°.

Of angles near 90° the tangents, both natural and logarithmic, change very fast and at rapidly varying rates. So of the cotangents of angles near 0°.

Of angles near 0° the logarithmic tangents change very fast and at rapidly varying rates. So of the logarithmic cotangents of angles near 90°.

These difficulties may be met in three ways:

1. For an angle near 90° use the half angle. [III pr. 4 p. 54

2. In the interpolation to find the function of a given angle, use two, three, or more orders of differences.

E.g., to find log-sin $10'\,30''$:

$$
\begin{array}{r|c|c|c|c}
\text{log-sin } 10' = 7.463726 & & & & \\
 & 41392 & & & \\
11' = 7.505118 & & -3604 & & \\
 & 37788 & & 578 & \\
12' = 7.542906 & & -3026 & & -129 \\
 & 34762 & & 449 & \\
13' = 7.577668 & & -2577 & & \\
 & 32185 & & & \\
14' = 7.609853 & & & &
\end{array}
$$

and $\text{log-sin } 10'\,30'' = 7.463726 + \frac{1}{2}(41392) - \frac{1}{8}(-3604)$
$+ \frac{1}{16}(578) - \frac{5}{128}(-129) = 7.484914.$

By proportional parts, *i.e.*, by interpolation with first differences only, log-sin $10'\,30'' = 7.463726 + \frac{1}{2}(41392) = 7.484422$, an error of .000492.

3. For small angles the ratios of the sine and the tangent to the radius-measure of the angle are near unity and change slowly [II § 3]; and since $1^r = 206265''$ nearly, the ratios of the sine and tangent to the number of seconds in the angle are near the quotient 1 : 206265, whose logarithm is $\bar{6}.685575$.

Special tables give the logarithms of these ratios exactly,

and $\sin A'' = A \cdot (\sin A'' : A), \quad \tan A'' = A \cdot (\tan A'' : A),$

E.g., $\text{log-sin } 10'\,30'' = \log 630 + \bar{6}.685574$
$= 2.799341 + \bar{6}.685574$
$= \bar{3}.484915, \quad \text{or} \quad 7.484915.$

PROB. 3. TO FIND LOG($A+B$), LOG($A-B$) FROM LOG A, LOG B.

Write $\tan^2\theta = B : A$ *and* $\cos^2\phi = B : A$; *find* θ *and* ϕ;

then $\log(A+B) = \log A - 2\,\text{log-cos}\,\theta,$
$\log(A-B) = \log A + 2\,\text{log-sin}\,\phi.$

For $\because\ A \pm B = A \cdot (1 \pm \overline{B : A}),$
$\therefore\ A + B = A \cdot (1 + \tan^2\theta) = A \cdot \sec^2\theta = A : \cos^2\theta,$

and $A - B = A \cdot (1 - \cos^2\phi) = A \cdot \sin^2\phi.$

EXAMPLES.

1...4. From the table of natural functions, find:

1. sin 20°, 21°, 20° 10′, 20° 10′ 45″, 79° 18′ 25″, 157° 15′ 23″.
2. cos 20°, 21°, 20° 10′, 20° 10′ 45″, 79° 18′ 25″, 157° 15′ 23″.
3. tan 35°, 36°, 35° 15′, 35° 15′ 47″, 79° 58′ 35″, 125° 0′ 12″.
4. cot 35°, 36°, 35° 15′, 35° 15′ 47″, 79° 58′ 35″, 125° 0′ 12″.

5...8. From the table of logarithmic functions, find:

5. log-sin 20°, 21°, 20° 10′, 20° 10′ 45″, 79° 18′ 25″, 157° 15′ 23″.
6. log-cos 20°, 21°, 20° 10′, 20° 10′ 45″, 79° 18′ 25″, 157° 15′ 23″.
7. log-tan 35°, 36°, 35° 15′, 35° 15′ 47″, 79° 58′ 35″, 125° 0′ 12″.
8. log-cot 35°, 36°, 35° 15′, 35° 15′ 47″, 79° 58′ 35″, 125° 0′ 12″.

9...12. From the table of natural functions, find:

9. $\sin^{-1}$ .25882, .25910, .25900, .92794, .92805, .92800.
10. $\cos^{-1}$ .92794, .92805, .92800, .25910, .25882, .25900.
11. $\tan^{-1}$ .5022, .5059, .5035, .9217, .9271, $^{-}$.9250.
12. $\cot^{-1}$ .9217, .9271, .9250, .5022, .5059, $^{-}$.5035.

13...16. From the table of logarithmic functions, find:

13. log-$\sin^{-1}$ 8.580892, 8.584193, 8.582125, 9.999683.
14. log-$\cos^{-1}$ 8.580892, 8.584193, 8.582125, 9.999683 neg.
15. log-$\tan^{-1}$ 8.581208, 8.584514, 8.583125, 1.418790 neg.
16. log-$\cot^{-1}$ 7.581208, 8.584514, 8.583125, 1.418790 neg.

17, 18. From the special table for small angles, find:

17. log-sin 10′, 30′ 25″, 1° 8′ 13″; log-tan 15′, 20′ 17″, 1° 37′ 12″.
18. log-$\sin^{-1}$ $\bar{2}.175398$, $\bar{1}.652798$; log-$\tan^{-1}$ $\bar{2}.564179$, $\bar{3}.196427$.
19. Given log-sin 17′, 18′, 19′, 20′, find log-sin 18′ 35″ by interpolation, using third differences.
20. Find A + B, A − B, given log A, log B = 3.838849, 2.792392; 1.826204, .690285; 2.908497, 1.110724.

*§ 2. THE GENERAL TRIANGLE.

Three directed lines lying in the same plane form a triangle whose sides are the three segments cut one from each line by the other two lines, and whose angles are the supplements of the three angles formed by the three lines taken two and two in order.

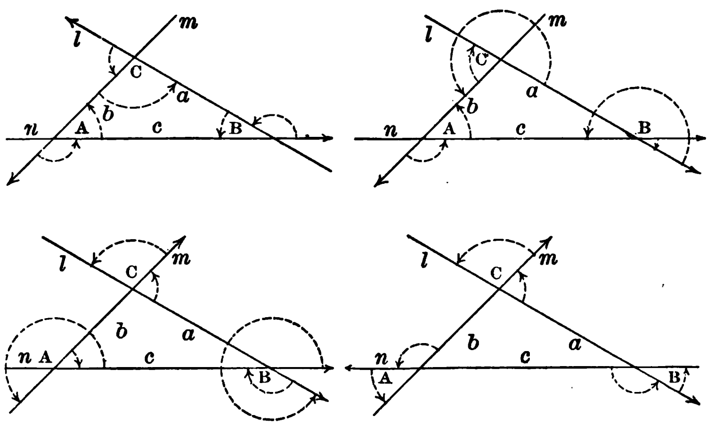
Let l, m, n be any three directed lines lying in the plane of the paper and meeting in the points A, B, C;

then the three sides a, b, c of the triangle ABC are the segments BC, CA, AB;

the three angles A, B, C are the supplements of the $\angle$s mn, nl, lm;

$$\angle mn + \angle nl + \angle lm = \angle mm = \text{four right angles,}$$

and $$\angle \text{A} + \angle \text{B} + \angle \text{C} = \text{two right angles.}$$



By the reversal of the lines l, m, n, seven other triangles are formed, eight in all from the same geometric lines:

$$lmn,\ {}^-lmn,\ l^-mn,\ lm^-n,\ l^-m^-n,\ {}^-lm^-n,\ {}^-l^-mn,\ {}^-l^-m^-n,$$

whereof the figures above show the first two and the last two; and by reading the lines in reverse order, still eight more are formed.

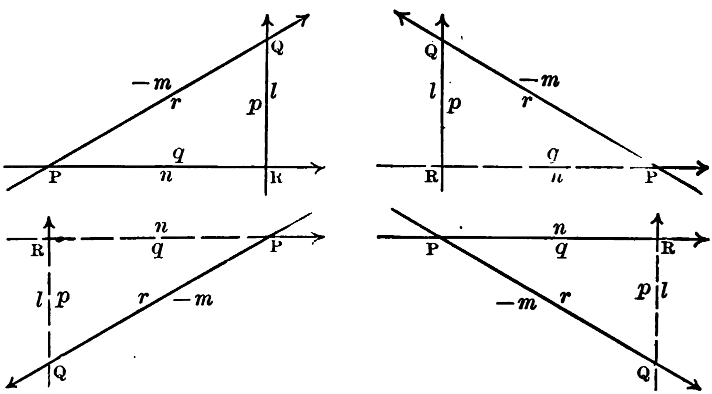
In the discussion of the general relations between the sides and angles of the triangle lmn the lines l, ^-m, n are best used when the angles A, B are involved, l, m, ^-n when B, C are involved, ^-l, m, n when C, A are involved; *i.e.*, the discussion is made symmetric by taking the triangle lmn as the type, using that line for initial line which joins the vertices of the two angles considered; and reversing the line before it, taken in the order l, m, n, l. Then two of the angles of this figure are identical with two angles of the triangle lmn, the third angles are supplementary, and the lines used (sides and perpendiculars) have the same directions as those used in the definitions of the trigonometric functions.

E.g., let the lines l, m, n form a right triangle PRQ;

wherein ∠s P, R, Q ≡ sup. ∠s mn, nl, lm,

and sides p, r, q ≡ segments RQ, QP, PR, on l, m, n.

In the figure use ^-m in place of m; then these same parts are ∠s $n\,^-m$, nl, ^-ml and segments RQ, PQ, PR on l, ^-m, n;



and the functions of P are read directly from the figure,

i.e., sin P $= p : r$, cos P $= q : r$, tan P $= p : q$ [df. trig. funcs.

and ∵ ∠s P, Q are complementary,

∴ the functions of Q are the co-functions of P,

i.e., sin Q $= q : r$, cos Q $= p : r$, tan Q $= q : p$. [II th. 8

The functions of Q may also be read from the figures by looking at them through the paper from the back, so that the lines may be properly directed.

§ 3. SOLUTION OF RIGHT PLANE TRIANGLES.

Three parts, always including a side, are sufficient to determine a plane triangle; and the *solution* of a triangle consists in finding the three unknown parts from the three that are given. In a right triangle, besides the right angle either two sides, or an oblique angle and a side, must be given.

The following principles relating to right triangles are here restated for convenience of reference:

1. The square of the hypothenuse is the sum of the squares of the two sides. [geom.

2. The sum of the two oblique angles is a right angle.

3. The product of the hypothenuse by the sine of an oblique angle is the opposite side. [df. sin

4. The product of the hypothenuse by the cosine of an oblique angle is the adjacent side. [df. cos

5. The product of a side by the tangent of the adjacent oblique angle is the opposite side. [df. tan

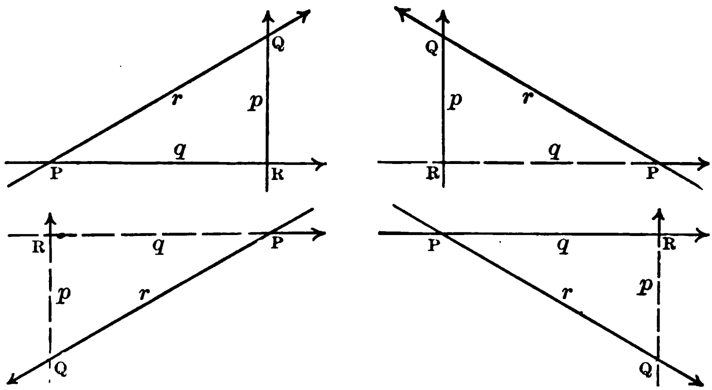
PROB. 4. TO SOLVE A RIGHT PLANE TRIANGLE.

Form three equations that express the values of the three parts sought in terms of two known parts, ignoring the right angle.

CHECK: *form an equation between the three computed parts, or between the part or parts last found and the given parts, in such manner that an error in the computation of any part makes the check a false equation.*

In no case may the same parts be used in the same way in the solution and the check.

Let PQR be any right triangle, P the base angle, Q the vertical angle, R the right angle, p, q, r the perpendicular, base, and hypothenuse.



(*a*) *Given* r, P, *the hypothenuse and an oblique angle:*

then $\quad$ $\text{Q} = 90° - \text{P}, \quad q = r \cdot \cos \text{P}, \quad p = r \cdot \sin \text{P}.$

Checks: $\text{Q} = \tan^{-1}(q : p), \qquad p^2 = (r + q)(r - q).$

(*b*) *Given* q, P, *a side and an oblique angle:*

then $\quad$ $\text{Q} = 90° - \text{P}, \quad r = q : \cos \text{P}, \quad p = q \cdot \tan \text{P}.$

Checks: $\text{Q} = \cos^{-1}(p : r), \qquad p^2 = (r + q)(r - q).$

(*c*) *Given* r, q, *the hypothenuse and a side:*

then $\quad$ $\cos \text{P} = q : r, \quad \text{Q} = 90° - \text{P}, \quad p = q \cdot \tan \text{P}.$

Checks: $\text{Q} = \cos^{-1}(p : r), \qquad p^2 = (r + q)(r - q).$

(*d*) *Given* q, p, *the two sides about the right angle:*

then $\quad$ $\tan \text{P} = p : q, \quad \text{Q} = 90° - \text{P}, \quad r = q : \cos \text{P}.$

Checks: $\cos \text{Q} = p : r, \qquad p^2 = (r + q)(r - q).$

E.g., if $r = 125, \quad \text{P} = 40°$:

then $\quad$ $\text{Q} = 90° - 40° = 50°$, and :

BY NATURAL FUNCTIONS.		BY LOGARITHMIC FUNCTIONS.	
.76604	.64279	$\bar{1}.884254$	$\bar{1}.808067$
125	125	2.096910	2.096910
383020	321395	1.981164	1.904977
153208	128558	39	32
76604	64279	45)250(6	54)450(8
$95.755 = q$.	$80.349 = p$.	$q = 95.756$.	$p = 80.348$.

check:		*check:*
80.349)95.755(1.19174		1.981164
nat-tan^{-1} 1.19174 = 50°.		1.904977
80.349	220.755	.076187
80.349	29.245	log-tan^{-1} .076187 = 50°.
723141	1103775	log 220.755 = 2.343910
321396	883020	log 29.245 = 1.466052
241047	441510	3.809962
6427920	1986795	log 80.349 = 1.904981
6455.962	441510	2
	6455.980	3.809962

The results of the work, by natural functions and by logarithms, as found above, do not quite agree. The discrepancies arise from the limitations and consequent defects of the tables. For the same reason the checks are not perfect.

[O. W. J. alg. V § 5, IX pr. 9

Neither of the checks is itself complete.

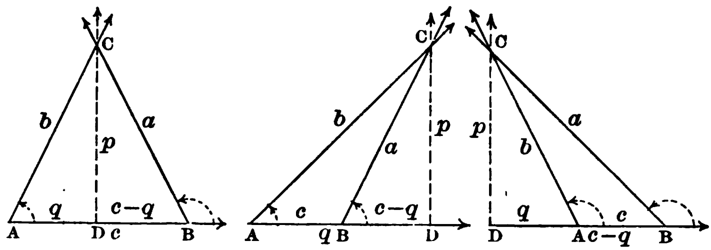
ISOSCELES TRIANGLES. If from the vertex of an isosceles triangle a perpendicular fall upon the base, the perpendicular divides the triangle into two equal right triangles.

Solve one of these right triangles.

OBLIQUE TRIANGLES. In general a perpendicular may fall from a vertex of an oblique triangle to the opposite side in such manner that one of the right triangles thus formed shall contain two of the three known parts, and the other right triangle shall contain one of them.

Solve these right triangles in order and so combine the parts as to find the parts sought in the given triangle.

If the three sides be given, drop a perpendicular p from C to the base c at D, dividing c into two segments AD, DB;
and let $q,\ c-q \equiv$ AD, DB;



then $\because\ p^2+q^2=b^2,\quad p^2+(c-q)^2=a^2,$

$\therefore\ 2cq-c^2=b^2-a^2;$

and $\quad q=[b^2+c^2-a^2]:2c,\quad c-q=[c^2+a^2-b^2]:2c.$

Solve the right triangles as before.

ANGLES NEAR 90° OR NEAR 0°.

1. *If* P *be near* 90°, *use the formulæ below.*

$$\tan\tfrac{1}{2}\text{P}=\sqrt{[(r-q):(r+q)]},\qquad \text{[II th. 14}$$
$$p=(r+q)\tan\tfrac{1}{2}\text{P}.$$

The formulæ assume that q, r are given; but if p, r be given, q may be computed; if P be small, the formulæ serve to find Q.

2. *If* P *be near* 0°, *use the table for small angles.*

E.g., if $q=858,\quad p=9.27$:

then $\quad \log 9.27 \ =\ .967080$

$\log 858 \ = \underline{2.933487-}$

$\log\tan \text{P}'' = \bar{2}.033593 \qquad \log\tan\text{P}''=\bar{6}.685591+\log \text{P}.$

$\underline{\bar{6}.685592-}$

$\log \text{P} = 3.348001$

$\text{P}'' = 2228''.4 = 37'\,8''.4.$

$\sin \text{P}'' = p:r.$

$$\log \sin P'' = \log p - \log r = \bar{6}.685566 + \log P.$$

$$\begin{array}{rlrl} \log P = & 3.348001 & \log p = & .967080 \\ & \underline{\bar{6}.685566+} & \log \sin P'' = & \underline{\bar{2}.033567-} \\ \log \sin P'' = & \bar{2}.033567 & \log r = & 2.933513 \\ & & r = & 858.051. \end{array}$$

So, if $P'' = 33'\,12''$, $p = 52.75$:

then $P = 1992''$, $\log \sin P'' = \bar{6}.685568 + \log P$.

$$\begin{array}{rlrl} \log P = & 3.299289 & \log p = & 1.722222 \\ & \underline{\bar{6}.685568+} & \log \sin P'' = & \underline{\bar{3}.984857-} \\ \log \sin P'' = & \bar{3}.984857 & \log r = & 3.737365 \\ & & r = & 5462.16 \end{array}$$

$$\begin{array}{rlrl} \log P = & 3.299289 & \log p = & 1.722222 \\ & \underline{\bar{6}.685588+} & \log \tan P'' = & \underline{\bar{3}.984877-} \\ \log \tan P'' = & \bar{3}.984878 & \log q = & 3.737345 \\ & & q = & 5461.92. \end{array}$$

EXAMPLES.

1...4. Solve the right triangles (preferably by aid of natural functions), and check the work, given:

1. $r = 36.3$, $P = 50°$. [40°, 27.81, 23.33
2. $q = 29.28$, $Q = 37°\,12'$. [52° 48′, 48.43, 38.57
3. $r = 125$, $p = 105$. [57° 8′ 24″, 32° 51′ 36″, 67.82
4. $q = 29.275$, $p = 39.07$. [53° 9′ 20″, 36° 50′ 40″, 48.82

5, 6. Solve the right triangles by aid of the table for small angles, and check the work, given:

5. $r = 37.09$, $p = .379$.
6. $r = 1311$, $P = 89°\,18'$.

7...9. Solve the isosceles triangles, and check the work, given:

7. base = 26.13, side = 127.8.
8. base = 231.1, base angle = 27° 19′.
9. side = 49.25, vertical angle = 57° 33′.

10...13. Solve the oblique triangles, and check the work, given:

10. $b = 52.01$, $A = 61° 8'$, $C = 35° 11'$.

11. $b = 2.054$, $c = 5.609$, $A = 97° 48'$

12. $a = 511.2$, $b = 247.8$, $c = 301.2$.

13. $a = 27$, $b = 64.98$, $A = 51° 24'$.

14. Find the ratio of the areas of two regular decagons, the one inscribed in, and the other circumscribed about, the same circle. [.9045

So, of two regular heptagons.

So, of two regular n-gons.

So, of a regular n-gon inscribed and a regular m-gon circumscribed.

15. In a circle of unit radius a regular pentagon is inscribed; find its side, its apothem, and its area.

So for a regular n-gon inscribed in a circle of radius r.

16. Find the angle at which the lateral face of a pyramid is inclined to the base, the faces being equilateral triangles and the base a square: thence find the diedral angle of a regular octaedron. [54° 44′ 8″, 109° 28′ 16″

17. In a regular tetraedron whose edge is unity, find the diedral angle of an edge, the perpendicular from the vertex to the base, and the distance apart of two opposite edges. [70° 31′ 44″, .8165, .7071

So in a regular pentagonal pyramid whose lateral edge is three times an edge of the base.

18. If two circles, of radii r, r', touch externally, and if θ be the angle between their common tangents, find the value of $\sin\theta$ in terms of r, r'.

§ 4. GENERAL PROPERTIES OF PLANE TRIANGLES.

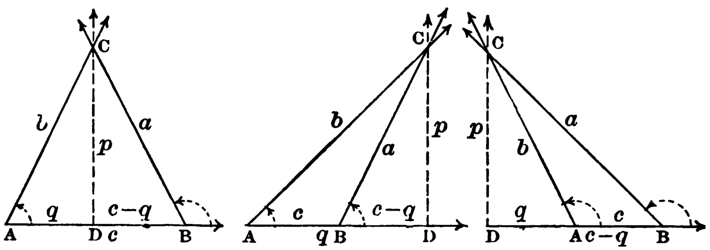
LAW OF COSINES.

Theor. 1. *In any plane triangle* ABC

$$\cos A = \frac{b^2+c^2-a^2}{2bc},\quad \cos B = \frac{c^2+a^2-b^2}{2ca},\quad \cos C = \frac{a^2+b^2-c^2}{2ab}.$$

For, drop a perpendicular p from C to the base c at D, dividing c into two segments AD, DB;

and let q, $c-q \equiv$ AD, DB;



then $\because\ b^2 = p^2 + q^2,\ a^2 = p^2 + (c-q)^2 = b^2 + c^2 - 2cq$;

and $\quad q = b \cos A$,

$\therefore\ a^2 = b^2 + c^2 - 2bc \cos A$,

$\therefore\ \cos A = (b^2 + c^2 - a^2) : 2bc$;

and so for cos B, cos C. Q. E. D.

Cor. 1. $\sin \frac{1}{2}A = \sqrt{[(s-b)(s-c) : bc]} \cdots$ $[s = \frac{1}{2}(a+b+c)$

For $\because\ 2\sin^2 \frac{1}{2}A = 1 - \cos A$ [II th. 13 cr.

$$= 1 - [(b^2 + c^2 - a^2) : 2bc]$$

$$= [a^2 - (b-c)^2] : 2bc$$

$$= (a-b+c)(a+b-c) : 2bc$$

$$= 4(s-b)(s-c) : 2bc,$$

$\therefore\quad \sin \frac{1}{2}A = \sqrt{[(s-b)(s-c) : bc]}$;

and so for $\sin \frac{1}{2}B$, $\sin \frac{1}{2}C$. Q. E. D.

COR. 2. $\cos\frac{1}{2}A = \sqrt{[s(s-a) : bc]} \cdots$.

For $\because\ 2\cos^2\frac{1}{2}A = 1 + \cos A$ [th. 13 cr.

$$= 1 + [(b^2+c^2-a^2) : 2bc]$$
$$= [(b+c)^2 - a^2] : 2bc$$
$$= (b+c+a)(b+c-a) : 2bc$$
$$= 4s(s-a) : 2bc,$$

$\therefore\ \cos\frac{1}{2}A = \sqrt{[s(s-a) : bc]}$;

and so for $\cos\frac{1}{2}B$, $\cos\frac{1}{2}C$. Q. E. D.

COR. 3. $\tan\frac{1}{2}A = \sqrt{[(s-b)(s-c) : s(s-a)]} \cdots$. [crs. 1, 2

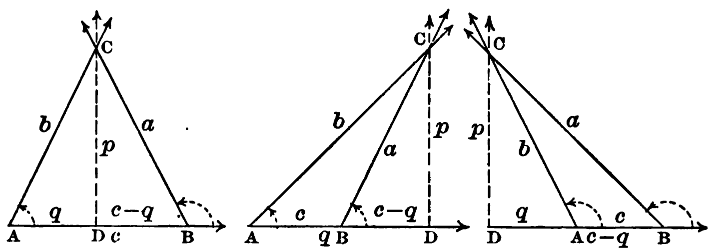
NOTE. In the limited triangle no angle is negative or greater than two right angles, no half-angle is negative or greater than a right angle; and the radicals of crs. 1–3 are all positive.

LAW OF SINES.

THEOR. 2. *In any plane triangle* ABC

$a : b : c = \sin A : \sin B : \sin C$;

For draw DC ordinate of C as to AB;



then $\because$ DC = AC sin A = b sin A, [df. sin

and DC = BC sin sup. B = a sin B,

$\therefore\ b \sin A = a \sin B$,

$\therefore\ a : b = \sin A : \sin B$. Q. E. D. [th. prop.

So $c : a = \sin C : \sin A$, [sym.

$\therefore\ a : b : c = \sin A : \sin B : \sin C$. Q. E. D.

COR. 1. $(a+b):c = \cos\frac{1}{2}(A-B) : \sin\frac{1}{2}C$,

$(a-b):c = \sin\frac{1}{2}(A-B) : \cos\frac{1}{2}C$.

For $\because\ a:b:c = \sin A : \sin B : \sin C$, [th. 2

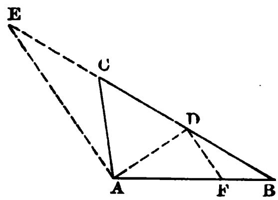
$\therefore\ (a+b):c = (\sin A + \sin B) : \sin C$

$= 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B) : 2\sin\frac{1}{2}C\cos\frac{1}{2}C$

$= \cos\frac{1}{2}(A-B) : \sin\frac{1}{2}C$. Q. E. D. [$\sin\frac{1}{2}(A+B) = \cos\frac{1}{2}C$

So $(a-b):c = \sin\frac{1}{2}(A-B) : \cos\frac{1}{2}C$.

GEOMETRIC PROOF. On BC take D, E such that CD, CE = CA;
then BD $= a-b$, BE $= a+b$.



The reader may prove that $\angle$ AEC $= \frac{1}{2}$C, that $\angle$ DAE is right, that $\angle$ BAD $= \frac{1}{2}$(A − B); and may then apply the law of sines to the triangles BAD and BAE.

COR. 2. $\tan\frac{1}{2}(A-B) = (a-b):(a+b)\cdot\cot\frac{1}{2}C$. [cr. 1

GEOMETRIC PROOF. In the figure of cr. 1 draw DF parallel to EA and therefore perpendicular to AD;

then $$\tan\tfrac{1}{2}(A-B) = \frac{DF}{AD} = \frac{DF}{EA}\cdot\frac{EA}{AD} = \frac{BD}{BE}\cot\tfrac{1}{2}C = \frac{a-b}{a+b}\cot\tfrac{1}{2}C.$$

EXAMPLES.

1...4. In any plane triangle ABC, show that:

1. $s:c = \cos\frac{1}{2}A\cos\frac{1}{2}B : \sin\frac{1}{2}C$. [$s = \frac{1}{2}(a+b+c)$
2. $(s-a):c = \cos\frac{1}{2}A\sin\frac{1}{2}B : \cos\frac{1}{2}C$.
3. $(s-b):c = \sin\frac{1}{2}A\cos\frac{1}{2}B : \cos\frac{1}{2}C$.
4. $(s-c):c = \sin\frac{1}{2}A\sin\frac{1}{2}B : \sin\frac{1}{2}C$.

5. If ABC be any plane triangle, express $\sin A$, $\cos A$, $\tan A$, and $\cos A + \cos B \cos C$, in functions of B, C.

6...12. Prove that:

6. $a \cos B + b \cos A = c$, $a \cos B - b \cos A = (a^2 - b^2) : c$.

7. $a \cos B \cos C + b \cos C \cos A + c \cos A \cos B$
$= a \sin B \sin C = b \sin C \sin A = c \sin A \sin B$.

8. $(a - b) \cot \frac{1}{2} C + (c - a) \cot \frac{1}{2} B + (b - c) \cot \frac{1}{2} A = 0$.

9. $a \cos A + b \cos B = c \cos (A - B)$, $b^2 \cos^2 C - c^2 \cos^2 B = b^2 - c^2$.

10. $a \cos A + b \cos B + c \cos C = 2 a \sin B \sin C = \cdots$.

11. $a \sin (B - C) + b \sin (C - A) + c \sin (A - B) = 0$.

12. $a \sin \frac{1}{2}(B - C) \sec \frac{1}{2} A + b \sin \frac{1}{2}(C - A) \sec \frac{1}{2} B$
$+ c \sin \frac{1}{2}(A - B) \sec \frac{1}{2} C = 0$.

§ 5. SOLUTION OF OBLIQUE PLANE TRIANGLES.

PROB. 5. TO SOLVE AN OBLIQUE PLANE TRIANGLE.

Apply such of the formulæ of ths. 1, 2 *as serve to express the value of each of the three unknown parts in terms of three known parts.*

CHECK: *form an equation involving the three computed parts and such of the given parts as may be necessary.*

In no case may the same parts be used in the same way in the solution and the check.

(*a*) *Given* A, B, *c*, *two angles and a side:*

then $C = 180° - (A + B)$, [geom.

$a = \sin A \cdot c : \sin C$, $b = \sin B \cdot c : \sin C$. [th. 2

The formulæ give one value, and but one, for each part.

(*b*) *Given a*, *b*, C, *two sides and the included angle:*

then $\tan \frac{1}{2} (A - B) = (a - b) : (a + b) \cdot \cot \frac{1}{2} C$, [th. 2 cr. 2

$\frac{1}{2} (A + B) = 90° - \frac{1}{2} C$,

$\frac{1}{2} (A + B) + \frac{1}{2} (A - B) = A$, $\frac{1}{2} (A + B) - \frac{1}{2} (A - B) = B$,

$c = \sin C \cdot a : \sin A$.

In the limited triangle the formulæ give one value, and but one, for each part.

(*c*) *Given a, b, c, the three sides:*

Apply the formulæ of th. 1.

In the limited triangle the formulæ give one value, and but one, for each part.

For the computation of all the angles, the formulæ of cr. 1 use nine different logarithms, those of cr. 2 use ten different logarithms, those of cr. 3 use seven different logarithms; the formulæ of cr. 3 are therefore preferred; they may be put in the form:

$$\tan \tfrac{1}{2}\text{A} = \frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

$$\tan \tfrac{1}{2}\text{B} = \frac{1}{s-b}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

$$\tan \tfrac{1}{2}\text{C} = \frac{1}{s-c}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

(*d*) *Given a, b, A, two sides and an angle opposite one of them:*

then $\quad \sin \text{B} = b \cdot \sin \text{A} : a,$

$$\text{C} = 180° - (\text{A} + \text{B}), \qquad \text{[geom.}$$

$$c = \sin \text{C} \cdot a : \sin \text{A}.$$

The formulæ leave the parts in doubt; for the same value of sin B belongs to two supplementary angles, so that in general B may be an acute or an obtuse angle; whence two values each for C, c, and two triangles.

But this is limited by the conditions, that the greater side of a triangle lies opposite the greater angle, and that a triangle can have but one obtuse angle.

If $a \not< b$, B is not obtuse; and if A be obtuse, B is acute.

The shortest length possible for a is the ordinate p, of C.

For $\because\ b \sin \text{A} = a \sin \text{B} = p,$

$\therefore$ if $a < p$, then $\sin \text{B} > 1$, which is impossible.

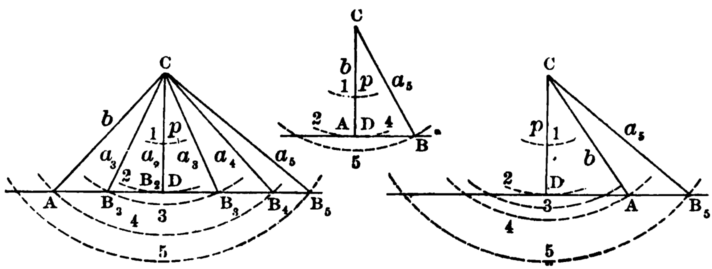
There are five cases as shown below:

1. *a shorter than p*:

There is no triangle.

2. *a just as long as p*:

If A be $\left\{\begin{matrix}\text{acute,}\\ \text{not acute,}\end{matrix}\right.$ there is $\left\{\begin{matrix}\text{one}\\ \text{no}\end{matrix}\right.$ triangle.



3. *a longer than p, but shorter than b*:

If A be $\left\{\begin{matrix}\text{acute,}\\ \text{obtuse,}\end{matrix}\right.$ there are $\left\{\begin{matrix}\text{two}\\ \text{no}\end{matrix}\right.$ triangles.

4. *a just as long as b*:

If A be $\left\{\begin{matrix}\text{acute,}\\ \text{not acute,}\end{matrix}\right.$ there is $\left\{\begin{matrix}\text{one}\\ \text{no}\end{matrix}\right.$ triangle.

5. *a longer than b*:

There is one triangle.

NOTE. The side c may be found by logarithms directly from the parts a, b, c, as follows:

$$\because\ c^2 = a^2 + b^2 - 2ab\cos\text{C}, \qquad [\text{th. } 1$$

$$\therefore\ c^2 = a^2 + b^2 - 2ab\,(2\cos^2\tfrac{1}{2}\text{C} - 1) \qquad [\text{II th. 13 cr.}$$

$$= a^2 + b^2 + 2ab - 4ab\cos^2\tfrac{1}{2}\text{C}$$

$$= (a+b)^2\,[1 - 4ab : (a+b)^2 \cdot \cos^2\tfrac{1}{2}\text{C}].$$

Find a *subsidiary angle* θ such that

$$\sin^2\theta = 4ab : (a-b)^2 \cdot \cos^2\tfrac{1}{2}\text{C};$$

then $\because\ c^2 = (a+b)^2\,(1 - \sin^2\theta) = (a+b)^2\cos^2\theta.$

$$\therefore\ c = (a+b)\cos\theta. \qquad \text{Q.E.F.}$$

EXAMPLES.

1...11. Solve the triangles (preferably by aid of logarithmic functions), and check the work, given:

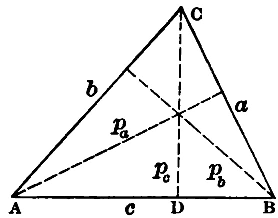
1. $a = 25.3$, $b = 136$, $C = 98° 15'$.
[$10° 9' 58''$, $71° 35' 2''$, 141.86
2. $A = 34°$, $B = 95°$, $c = 13.89$. [$51°$, 9.995, 17.805
3. $a = 127$, $b = 64.9$, $c = 152.16$.
[$55° 19' 39''$, $24° 51' 7''$, $99° 49' 14''$
4. $a = 18$, $b = 20$, $A = 55° 24'$. [$66° 8' 54''$, $58° 27' 6''$, 18.64,
or $113° 51' 6''$, $10° 44' 54''$, 4.08
5. $a = 10$, $b = 20$, $A = 30°$. [$90°$, $60°$, 17.32
6. $a = 16$, $b = 20$, $A = 86° 40'$.
7. $a = 20$, $b = 20$, $A = 47° 9'$. [$47° 9'$, $85° 42'$, 27.2
8. $a = 24$, $b = 20$, $A = 37° 36'$.
[$30° 33' 39''$, $111° 50' 21''$, 36.51
9. $a = 24$, $b = 20$, $A = 120°$. [$46° 11' 39''$, $13° 48' 21''$, 6.61
10. $a = 20$, $b = 20$, $A = 135°$.
11. $a = 16$, $b = 20$, $A = 150°$.
12. By the table for small angles solve the triangles, given:
$a = 127$, $b = 254$, $c = 380$; $a = 2000$, $b = 1999$, $A = 91°$.

13...15. Solve the triangles without tables, given:

13. $a = 1$, $b = \sqrt{2}$, $A = 30°$; $a : b : c = 2 : \sqrt{6} : (1 + \sqrt{3})$;
14. $a = 1$, $b = 1 + \sqrt{5}$, $A = 18°$; $a = 1$, $b = 1 + \sqrt{3}$, $A = 15°$;
15. $a : b : c = (m^2 + m + 1) : (2m + 1) : (m^2 - 1)$.
16. Solve for c the quadratic equation $a^2 = b^2 + c^2 - 2bc \cos A$;
find the sum and the product of the two roots;
find the conditions for real and separate, real and equal, and imaginary roots;
find the conditions for positive, zero, and negative roots;
interpret all these results by the figures on page 62.

§ 6. AREA OF A PLANE TRIANGLE.

PROB. 6. TO FIND THE AREA OF A PLANE TRIANGLE, AND THE ORDINATES OF THE VERTICES AS TO THE OPPOSITE SIDES.



Let ABC be any triangle, let K stand for its area, and p_a, p_b, p_c, for the ordinates of A, B, C as to a, b, c.

(a) *Given two sides and the included angle:*

For the area, multiply half the product of any two sides by the sine of the included angle.

For the ordinate of either vertex, multiply an adjacent side by the sine of the angle included by this side and the side opposite the given vertex.

For $\quad p_c = b \sin A$,

and $\quad K = \frac{1}{2} p_c \cdot c = \frac{1}{2} bc \sin A$; and so for the rest.

(b) *Given one side and the angles:*

For the area, multiply half the square of any side by the sines of the adjacent angles, and divide the product by the sine of the opposite angle.

For the ordinate, multiply the opposite side by the sines of the adjacent angles, and divide the product by the sine of the angle.

For $\quad \because K = \frac{1}{2} bc \sin A$,

and $\quad b = \sin B \cdot c : \sin C$, [th. 2

$\quad \therefore K = \frac{1}{2} c^2 \sin A \sin B : \sin C$,

and $\quad p_c = 2K : c = c \sin A \sin B : \sin C$; and so for the rest.

(c) *Given the three sides:*

For the area, from half the sum of the sides subtract each side separately, find the continued product of these remainders and the half-sum, and take the square root of this product.

For an ordinate, divide twice the root above found by the side on which the ordinate falls.

For $\because \text{K} = \frac{1}{2} ab \sin \text{C}$,

and $\sin \text{C} = 2 \sin \frac{1}{2}\text{C} \cos \frac{1}{2}\text{C}$

$$= 2\sqrt{[(s-a)(s-b):ab]}\cdot\sqrt{[s(s-c):ab]}$$
$$= 2\sqrt{[s(s-a)(s-b)(s-c)]}:ab,$$

$$\therefore \text{K} = \sqrt{[s(s-a)(s-b)(s-c)]};$$

and $p_c = 2\text{K} : c$; and so for p_a, p_b.

Check: $\dfrac{1}{p_a} + \dfrac{1}{p_b} + \dfrac{1}{p_c} = \dfrac{s}{\text{K}}$.

EXAMPLES.

1...3. Find the areas of the triangles, and the ordinates, given:

1. $a = 12.5$, $b = 25$, $\text{C} = 38°$. [$\text{K} = 96.2$

2. $\text{A} = 38° 18'$, $\text{B} = 91° 28'$, $c = 39.5$. [$\text{K} = 628.8$

3. $a = 287$, $b = 8.262$, $c = 36.47$. [$\text{K} = 45.39$

4. The sides of a triangle are in arithmetic progression, and its area is three-fifths of that of an equilateral triangle of the same perimeter; find the ratio of the sides, and the largest angle.

5. Find an expression for the sum of the areas, if from the data, a, b, A, there be two triangles.

6...8. Prove the following values for K:

6. $\frac{1}{4}(a^2 \sin 2\text{B} + b^2 \sin 2\text{A})$, $abc \cos \frac{1}{2}\text{A} \cos \frac{1}{2}\text{B} \cos \frac{1}{2}\text{C} : s$,

7. $\frac{1}{2}(a^2 - b^2) \sin \text{A} \sin \text{B} : \sin(\text{A} - \text{B})$,

8. $s^2 : (\cot \frac{1}{2}\text{A} + \cot \frac{1}{2}\text{B} + \cot \frac{1}{2}\text{C})$, $\sqrt{abcs} \sin \frac{1}{2}\text{A} \sin \frac{1}{2}\text{B} \sin \frac{1}{2}\text{C}$.

9. In a right triangle PQR, let h be the ordinate of R as to r, then: $h^{-2} = p^{-2} + q^{-2}$, $r = \sqrt{[(p-q)^2 + 4\text{K}]}$.

*§ 7. INSCRIBED, ESCRIBED, AND CIRCUMSCRIBED CIRCLES.

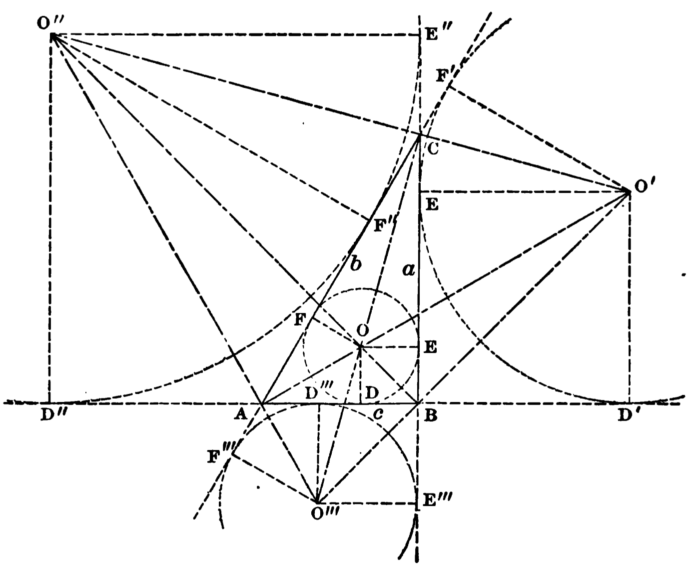
Prob. 7. To find the radii of the circles inscribed in, escribed, and circumscribed about, any triangle.

For the radius of the inscribed circle, divide the area by half the perimeter.

For the radius of an escribed circle, divide the area by half the perimeter less the side beyond which the circle lies.

For the radius of the circumscribed circle, divide half of either side by the sine of the opposite angle.

For, let ABC be any triangle, and let $r \equiv$ radius of inscribed circle, r', r'', $r''' \equiv$ radii of escribed circles whose centres are O′, O″, O‴, and $R \equiv$ radius of circumscribed circle;



then $\because$ $K = \frac{1}{2} r (a + b + c) = rs$, [geom.

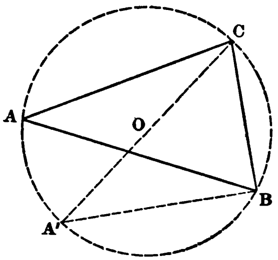
$\therefore$ $r = K : s$. Q. E. D.

So $\because$ $K = \frac{1}{2} r' (-a + b + c) = r' (s - a)$, [geom.

$\therefore$ $r' = K : (s - a)$; and so for r'', r'''. Q. E. D.

Checks: $\frac{1}{r} = \frac{1}{r'} + \frac{1}{r''} + \frac{1}{r'''}$, $K^2 = r \cdot r' \cdot r'' \cdot r'''$.

About ABC draw a circle and draw CA′, a diameter; join A′B;



then $\because$ A = A′, and $\angle$ A′BC is a right angle, [geom.
and CA′ = a : sin A′ = a : sin A,

$\therefore$ R = $\frac{1}{2}a$: sin A, ⋯. Q. E. D.

NOTE. a : sin A = b : sin B = c : sin C = 2 R.

EXAMPLES.

1…3. Find the radii of the inscribed, escribed, and circumscribed circles, and check the work, given:

1. $a = 12.7$ $b = 22.8$, $c = 33.9$.
2. A = 64° 19′ 8″, B = 100° 2′ 27″, $c = 51.25$.
3. $a = 136$, $b = 95.2$, C = 11° 37′.
4. In a right triangle, 2 R + $r = s$.
5. If R = 2 r, the triangle is equilateral.
6. In the ambiguous case the two values of R are equal.
7. The distances from the centre of the inscribed circle to the centres of the three escribed circles are equal to 4 R sin $\frac{1}{2}$A, ⋯, and to a sec $\frac{1}{2}$A, ⋯.
8. The square of the distance between the centres of the inscribed and circumscribed circles is $R^2 - 2Rr$.

9. Prove the equations:

$$r = (s - a) \tan \tfrac{1}{2} A,$$

$$r = s \tan \tfrac{1}{2} A \tan \tfrac{1}{2} B \tan \tfrac{1}{2} C,$$

$$r = a : (\cot \tfrac{1}{2} B + \cot \tfrac{1}{2} C), \quad r' = a : (\tan \tfrac{1}{2} B + \tan \tfrac{1}{2} C),$$

$$R = abc : 4K,$$

$$R = s : (\sin A + \sin B + \sin C),$$

$$r' + r'' + r''' - r = 4R; \quad rr' : r''r''' = \tan^2 \tfrac{1}{2} A,$$

$$K = 4Rr \cos \tfrac{1}{2} A \cos \tfrac{1}{2} B \cos \tfrac{1}{2} C,$$

$$R + r = R(\cos A + \cos B + \cos C),$$

$$4R \sin A \sin B \sin C = a \cos A + b \cos B + c \cos C.$$

§ 8. HEIGHTS AND DISTANCES.

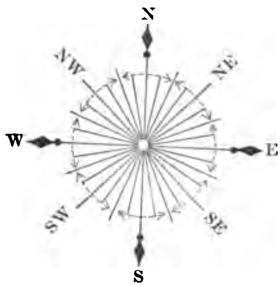
The plane of the *horizon* at any point is the plane that is tangent to the earth's surface, *i.e.*, to the surface of still water, at that point; it would therefore be perpendicular to the radius if the earth were a perfect sphere. The direction perpendicular to the horizon-plane is determined by the plumb-line and is the *vertical line*, and any plane containing this line is a *vertical plane*. Any plane parallel to the horizon-plane is a *horizontal plane* and may be determined by a spirit-level.

An angle lying in a horizontal plane is a *horizontal angle*, and an angle lying in a vertical plane is a *vertical angle*.

In surveying, the *bearing* of a point is the horizontal angle that the horizontal projection of the *line of sight*, from the observer to the point, makes with the north-and-south line through the point of observation; and the *angle of elevation* is the inclination of the line of sight to the horizontal plane; it is a vertical angle, and is called the *angle of depression* if the line of sight be below the plane of observation.

Ordinary field instruments measure horizontal and vertical angles only. By *distance* is meant the horizontal distance, unless otherwise named; and by *height* is meant the vertical distance of a point above or below the plane of observation.

Bearings at sea are given in *points* and *quarter-points*, counted from each of the eight cardinal points, two points each way.



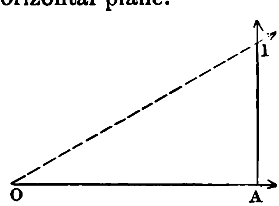
E.g., the points from north to east are: N., N. by E., N.N.E., N.E. by N., N.E., N.E. by E., E.N.E., E. by N., E.;

and the points and quarter-points are: N., N. $\frac{1}{4}$ E.; N. $\frac{1}{2}$ E., N. $\frac{3}{4}$ E., N. by E., N. by E. $\frac{1}{4}$ E., N. by E. $\frac{1}{2}$ E., N. by E. $\frac{3}{4}$ E., N.N.E.; N.E. by N. $\frac{3}{4}$ N., N.E. by N. $\frac{1}{2}$ N., ..., N.E.; N.E. $\frac{1}{4}$ E., N.E. $\frac{1}{2}$ E., ..., E.N.E.; E. by N. $\frac{3}{4}$ N., E. by N. $\frac{1}{2}$ N., ..., E.

PROB. 8. TO FIND THE HEIGHT ABOVE ITS BASE OF A VERTICAL COLUMN WHOSE BASE IS ACCESSIBLE.

(*a*) *Standing upon a horizontal plane:*

Let P be the top of the column, and A the point vertically below it on the horizontal plane.

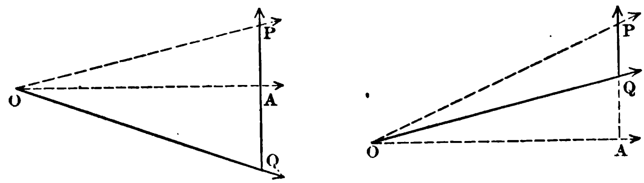


Measure any convenient distance OA *and the angle* AOP.

Solve the right triangle AOP *for* AP.

(*b*) *Standing upon an inclined plane:*

Let P be the top of the column, and Q the point vertically below it on the inclined plane.



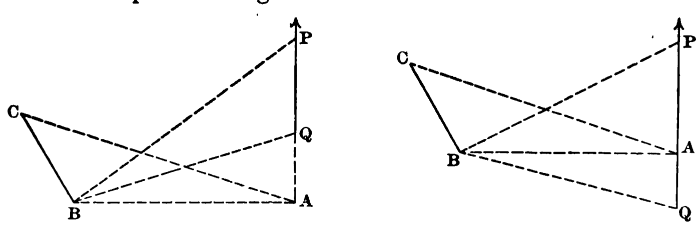
Measure any convenient distance OQ *along the plane, and the angles of elevation or depression* AOP, AOQ;

then $\angle\ \mathrm{QOP} = \mathrm{AOP} - \mathrm{AOQ}$, $\angle\ \mathrm{OPQ} = 90° - \mathrm{AOP}$.

Solve the oblique triangle OPQ *for* QP.

PROB. 9. TO FIND THE HEIGHT ABOVE ITS BASE, AND THE DISTANCE FROM THE OBSERVER, OF AN INACCESSIBLE BUT VISIBLE VERTICAL COLUMN.

Let P be the top of the column, Q the base, B the position of the observer, A the point vertically below P in the horizontal plane through B.



Take any other convenient point of observation C, *and measure the horizontal distance* BC, *the horizontal angles corresponding to* ABC, ACB, *and the vertical angles* ABP, ABQ.

Solve the horizontal oblique triangle corresponding to ABC *for* AB, *and the vertical right triangles* ABP, ABQ *for* AP, AQ;

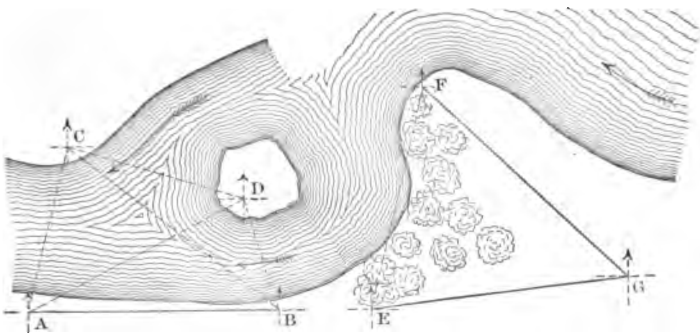
then $\mathrm{QP} = \mathrm{AP} - \mathrm{AQ}$.

If the observer be in the same horizontal plane as the base, the line BQ coincides with BA, and BAP is the only vertical triangle to be computed.

PROB. 10. TO FIND THE DISTANCE APART, AND THE BEARING, ONE FROM THE OTHER, OF TWO OBJECTS THAT ARE SEPARATED BY AN IMPASSABLE BARRIER.

(*a*) *Both objects accessible:*

Let E, F be the two objects, and G the point of observation.



Measure the distances GE, GF *and their bearings.*

From the oblique triangle EFG *compute* EF, ∠ GEF *and the bearing of* EF.

(*b*) *Both objects inaccessible:*

Let C, D be the two objects.

Measure any convenient distance AB *and the bearings of* AC, AD, BC, BD, *and compute in order such parts as are needed of the triangles* ABC, ABD, ACD.

This is the method of *triangulation* and AB is the *base line*.

EXAMPLES.

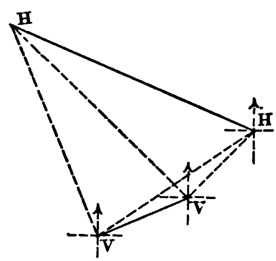
1. At 120 feet distance, and on a level with the foot of a steeple, the angle of elevation of the top is 62° 27′; find the height. [230.03 ft.

2. From the top of a rock 326 feet above the sea the angle of depression of a ship's hull is 25° 42′; find the distance of the ship. [677.38 ft.

3. A ladder $29\frac{1}{2}$ feet long standing in the street just reaches a window 25 feet high on one side of the street, and 23 feet high on the other side; how wide is the street?
[34.13 ft.

4. Find the distance across a river, if the base $AB = 475$ ft., $\angle A = 90°$, $\angle B = 57° 13' 20''$. [737.68 ft.

5. From a vessel two headlands were observed: the first bore N.N.W., and the second N.E. by E.; then sailing 12 miles



E.N.E., the first bore N.W. and the second N.E. Find the bearing and distance of one headland from the other.
[W.N.W. nearly, 33.575 miles.

6. From the top of a mountain $1\frac{1}{2}$ miles high, the *dip* of the sea-horizon (angle of depression of sky-and-water line) is $1° 34' 40''$; find the earth's diameter, and the distance of the sea-horizon.

7. What is the distance and the dip of the sea-horizon from the top of a mountain $2\frac{3}{4}$ miles high, the earth's mean radius being 3956 miles? [$2° 8' 8''$

8. If the dip of the sea-horizon be 1°, find the height of the mountain, and the distance of the sea-horizon.

9. Given the earth's equatorial radius 3962.76 miles, and the angle this radius subtends at the sun, $8''.81$; find the distance of the earth from the sun.
[92,780,000 miles nearly.

10. How far should a coin an inch in diameter be held from the eye to subtend an angle of 1°?

11. Given CA = 131 ft. 5 in., BC = 109 ft. 3 in.; $\angle$ C = 98° 34′; what is the distance AB? [183 ft.

12. From the top of a hill I observe two successive milestones in the plain below, and in a straight line before me, and find their angles of depression to be 5° 30′, 14° 20′; what is the height of the hill? [815.85 ft.

13. Two observers on the same side of a balloon, and in the same vertical plane with it, are a mile apart, and find the angles of elevation to be 17° and 68° 25′ respectively; what is its height? [1836 ft.

14. Two ships lying half a mile apart, each observes the angle subtended by the other ship and a fort to be, 56° 19′ and 63° 14′; find the distances of the ships from the fort. [2525, 2710 ft.

15. A privateer lies 12 miles S.W. of a harbor, and a merchantman leaves it in the direction E. by S., at the rate of 10 miles an hour; on what course, and at what rate, must the privateer sail in order to overtake the merchantman in an hour and a half? [E. by N. $\frac{1}{4}$ N. and 16 miles, nearly

16. Given [fig. p. 71] the base AB = 131$\frac{1}{2}$ yds., $\angle$ BAD = 50°, $\angle$ BAC = 85° 15′, $\angle$ DBC = 38° 43′, $\angle$ DBA = 94° 13′; what is the distance CD? Check the work by making two distinct computations from the data. [129.99

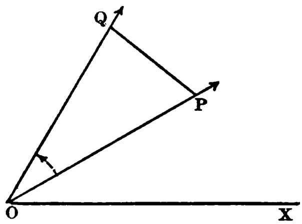
17. If A, B, C be three co-linear points, O any other point, and $a, b, c, l, m, n \equiv$ BC, CA, AB, OA, OB, OC, wherein $a + b + c = 0$; show that $al^2 + bm^2 + cn^2 = -abc$.

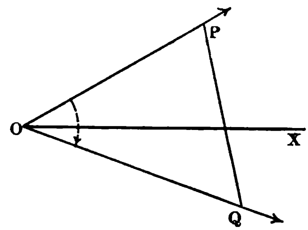
18. Three observers at the co-linear points A, B, C, of ex. 17, find the elevations of a balloon to be α, β, γ; show that the square of its height is

$$-abc : (a \cot^2 \alpha + b \cot^2 \beta + c \cot^2 \gamma).$$

*§ 9. POSITIVE AND NEGATIVE AREAS.

If a point move from P to Q along a straight line PQ, the radius vector from an origin O to the moving point sweeps





over a triangle whose area is $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right.$ when the angle POQ is $\left\{\begin{matrix}\text{positive;}\\ \text{negative;}\end{matrix}\right.$ for this area is half the product OP · OQ · sin POQ, wherein OP, OQ are both positive.

In the three theorems that follow, it is assumed that every motion of a $\left\{\begin{matrix}\text{point}\\ \text{line}\end{matrix}\right.$ is the limit of some motion made up of small $\left\{\begin{matrix}\text{translations along}\\ \text{rotations about}\end{matrix}\right.$ successive $\left\{\begin{matrix}\text{lines.}\\ \text{points.}\end{matrix}\right.$

Either motion is that of a point and a line through it, such that the point always slides along the line, while the line always swings about the point.

E.g., if a line roll round a circle, without sliding upon it, the line always swings about the point of contact, while the point of contact always slides along the tangent line.

The *area swept over by a segment* of a straight line is the algebraic sum of the areas of all the infinitesimal quadrilaterals and triangles passed over, from instant to instant, by the segment.

THEOR. 3. *If* PQR ··· TP *be any closed figure traced by the end of a radius vector, drawn from* O, *and varying in length if need be, the area of this figure is the area swept over by the radius vector; and is* $\left\{\begin{matrix}\textit{positive}\\ \textit{negative}\end{matrix}\right.$ *when the bounding line is traced in the* $\left\{\begin{matrix}\textit{positive}\\ \textit{negative}\end{matrix}\right.$ *direction of revolution.*

(*a*) *No reversals of motion of the vector*, as in the first figure, or *only one reversal*, as in the second figure :

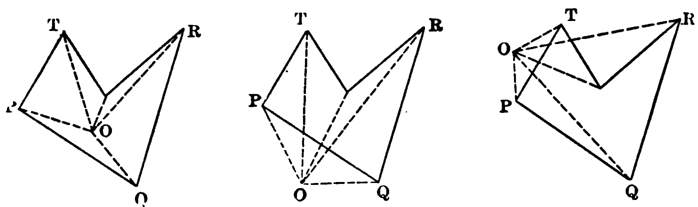
For ∵ there are no intermediate reversals, [hyp.

∴ the figure enclosed by the boundary is swept over once, and but once, by the vector, when it swings in the direction in which the path is traced ;

and ∵ all other figures swept over by the vector in one direction are also swept over in the other direction, and cancelled,

∴ the algebraic sum of the areas of all the figures swept over is the area of the figure enclosed by the boundary.

and this area is $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right.$ when the path is traced in the $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right.$ direction of rotation.



(*b*) *Intermediate reversals of motion*, as in the third figure :

For ∵ intermediate reversals occur in consecutive pairs in opposite directions,

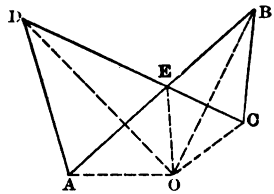
∴ if a point within the enclosure be swept over more than once, it is swept over an odd number of times so as to give an excess of just one passage in the forward direction ;

and ∵ every point without the enclosure is swept over, if at all, the same number of times in each direction ; so that any outside area that may be generated is cancelled,

∴ the algebraic sum of the areas of all the figures swept over is the area sought. Q. E. D.

NOTE 1. If the boundary cross itself, the figure is thus divided into two or more parts; the area of each part may be considered separately, and the area of the whole is the algebraic sum of the areas of the several parts.

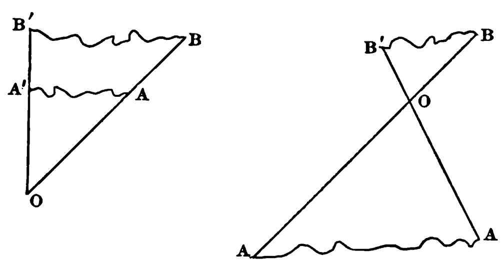
E.g., the area of the crossed quadrilateral ABCD is the alge-



braic sum of the areas of the positive triangle AED and the negative angle EBC, and has the sign of the larger.

NOTE 2. In adding two areas, any common boundary traversed in opposite directions may be erased.

COR. *If a segment* AB *of a vector* OB *swing about* O *as centre into the position* A'B', *the area of the figure swept over by this segment is the area of the figure* ABB'A', *bounded by the initial and terminal positions of the segment and the paths of its ends.*



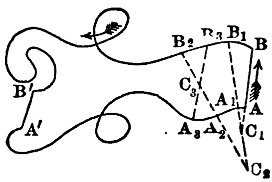
For $\because$ AB = OB − OA,

$\therefore$ the area K of the figure swept over by the segment AB is the area of the figure swept over by vector OB less the area of the figure swept over by vector OA,

$\therefore$ K = OBB' − OAA' = OBB' + OA'A = ABB'A'A. Q.E.D. [th. 3

THEOR. 4. *If two points* A, B *move* (forward or backward in any way) *along any paths* AA′, BB′ *to* A′, B′, *then the area swept over by the straight line* AB (varying in length if need be) *is the area of the figure* ABB′A′.

For let the motion of the generator AB be made up of infinitesimal rotations about successive instantaneous centres C_1, C_2, C_3, ⋯;



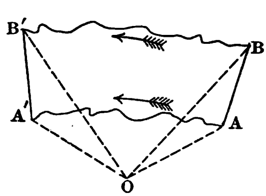
then ∵ AB sweeps over figures ABB_1A_1, $A_1B_1B_2A_2$, ⋯, [th. 3 cr.

and ∵ all the intermediate positions A_1B_1, A_2B_2, ⋯ of AB are common boundaries of these figures traced in opposite directions,

∴ the sum of all the areas swept over is the area of the figure bounded by the path ABB′A′A.

Q. E. D. [th. 3 nt. 2

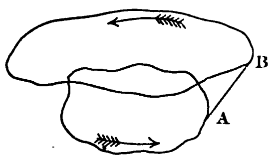
COR. 1. *The area swept over by any straight line*, AB, *is the sum of the excess of the area of the figure subtended* (from any origin) *by the path of the terminal point*, B, *over that subtended*

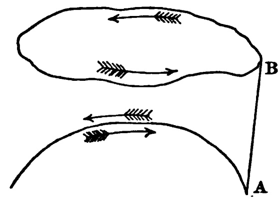


by the path of the initial point, A, *and the excess of the area of the triangle subtended by the initial line*, AB, *over that subtended by the terminal line*, A′B′;

i.e., $ABB'A'A = (OBB' - OAA') + (OAB - OA'B')$.

COR. 2. *If the generator,* AB, *return to its initial position, the area swept over is the excess of the area of the figure bounded by the path of the terminal point,* B, *over that of the figure bounded by the path of the initial point,* A.



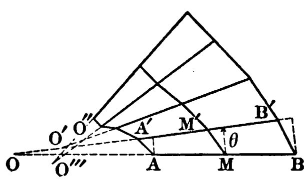


COR. 3. *If the generator,* AB, *return to its initial position, and the initial point,* A, *trace out the same path to and fro, then the area swept over is the area of the figure bounded by the path of the terminal point,* B.

THEOR. 5. *If a wheel be affixed to its axis at the mid-point, and if this wheel roll and slide in any way upon a plane while its axis remains parallel to the plane, the area swept over by the axis is the product of its length into the distance rolled by the wheel.*

For, let AB be the axis and M the mid-point;

let the axis turn about an instantaneous centre O, through an infinitesimal angle θ, and at the same time let the axis slide along its line an infinitesimal distance, to A′B′;



then $\because$ OA $\doteq$ OA′, OB $\doteq$ OB′, OM $\doteq$ OM′, $\sin\theta \doteq \theta$,

$\therefore$ area $ABB'A'$, $= OBB' - OAA'$, $\doteq \frac{1}{2}(OB^2 - OA^2)\cdot\theta$
$= \frac{1}{2}(OB - OA)(OB + OA)\cdot\theta = AB\cdot OM\cdot\theta$
$= AB\cdot$ the distance rolled by the wheel at M.

$\therefore$ the area swept over by any number of such successive rotations is the product of AB by the distance rolled by the wheel at M. Q. E. D.

Cor. 1. *If the wheel be affixed to its axis at any other point*, C, *and the axis turn through an angle*, α, *between its first and last positions, the area swept over is*

$$AB\cdot\textit{the distance rolled by the wheel at } C + AB\cdot CM\cdot\alpha.$$

For $\because$ in the infinitesimal rotation above,
area $AB\cdot OM\cdot\theta = AB\cdot(OC + CM)\cdot\theta$
$= AB\cdot$ the distance rolled by the wheel at $C + AB\cdot CM\cdot\theta$,

$\therefore$ the sum of all such rotations is $AB\cdot$ the distance rolled by the wheel at $C + AB\cdot CM\cdot\alpha$. [$\alpha = \theta + \theta' + \cdots$

Cor. 2. *If the axis return to its first position without making a complete revolution, the area swept over is* $AB\cdot$ *the distance rolled by the wheel affixed at any point* C. [$\alpha = 0$

EXAMPLES.

1. If A, B, C be fixed points on a line that turns in a plane through an angle α,

   then BC area $AB - AB$ area $BC = \frac{1}{2}AB\cdot BC\cdot CA\cdot\alpha$.

2. If the line in ex. 1 return to its first position:

   (a) without making a complete revolution,
   area $B = (AB$ area $C + BC$ area $A) : AC$;

   (b) after making a complete revolution,
   area $B + \pi\cdot AB\cdot BC = (AB$ area $C + BC$ area $A) : AC$.

3. If the chord AC, in ex. 1, slide round an oval, the area between the oval and the path of B is $\pi\cdot AB\cdot BC$.

4. Find the area of the curve traced by a point on the connecting rod of a piston and crank in one revolution; also the distance a small wheel attached at the same point would roll if a plane surface pressed against it.

AMSLER'S PLANIMETER.

Let the axis AB, above noted, be pivoted at A to an arm OA of fixed length that turns about a fixed centre O, so that A traces a fixed circle while B traces any path whatever;

let the wheel be affixed to AB at any point C, but let it be impossible for AB to sweep past OA so that AB, OA can take but one position for one position of B and, if A encircle O, AB also encircles O in the same direction:

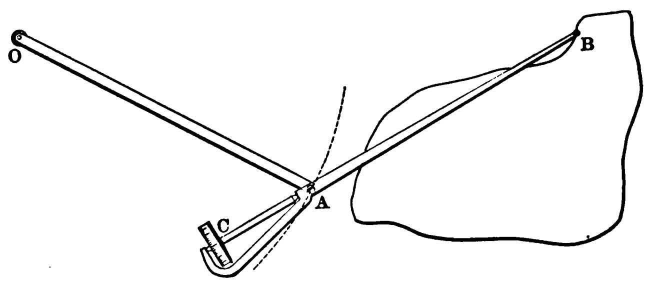
1. If A return to its first position without encircling O;

then ∵ A traces out the same path, to and fro,

∴ the area encircled by B is the area swept over by AB, [th. 4 cr. 3

i.e., the area is the product of the number of turns of the wheel into the constant area $2\pi r \cdot AB$, [th. 5 cr. 2

wherein r is the radius of the wheel.



2. If A encircle O counter-clock-wise;

then the area encircled by B is the area swept over by AB + the area of the circle OA, [th. 4 cr. 2

i.e., the area encircled by B is $2\pi r \cdot AB \cdot$ the number of turns of the wheel (positive or negative) $+ AB \cdot CM \cdot a + \pi \cdot OA^2$

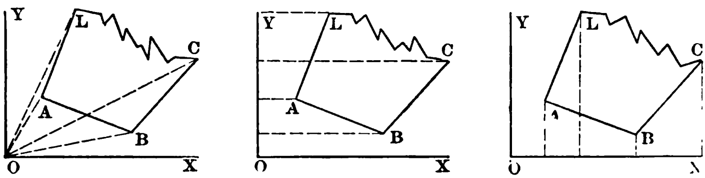
wherein a is 2π. [th. 5 cr. 1

The *constants of the planimeter* $2\pi r \cdot AB$, $\pi(2AB \cdot CM + OA^2)$ can be found once for all. The first is the area due to one turn of the wheel; the second is that due to the swinging of the arms OA, AB, about O.

§ 10. AREA OF A POLYGON.

PROB. 10. TO FIND THE AREA OF A POLYGON.

Let ABC ⋯ L be any polygon, K its area, $r_1, \theta_1, r_2, \theta_2, r_3, \theta_3, \cdots r_n, \theta_n$, and $x_1, y_1, x_2, y_2, x_3, y_3, \cdots x_n, y_n$ be the polar and the rectangular coordinates of A, B, C, ⋯ L.



(a) *By aid of triangles:*

Join OA, OB, OC, ⋯ OL;

then $$\text{K} = \tfrac{1}{2}\left[r_1 r_2 \sin(\theta_2 - \theta_1) + \cdots + r_n r_1 \sin(\theta_1 - \theta_n)\right] \quad \text{[th. 3}$$
$$= \tfrac{1}{2}\left[(x_1 y_2 - x_2 y_1) + \cdots + (x_n y_1 - x_1 y_n)\right].$$

(b) *By aid of trapezoids on the y-axis:*

Draw the ordinates of A, B, C, ⋯ *as to the y-axis;*

then $$\text{K} = \tfrac{1}{2}\left[(x_1 + x_2)(y_2 - y_1) + \cdots + (x_n + x_1)(y_1 - y_n)\right].$$

(c) *By aid of trapezoids on the x-axis:*

Draw the ordinates of A, B, C, ⋯ *as to the x-axis;*

then $$\text{K} = -\tfrac{1}{2}\left[(y_1 + y_2)(x_2 - x_1) + \cdots + (y_n + y_1)(x_1 - x_n)\right].$$

E.g., to find the area of a quadrilateral the coordinates of whose vertices are 1, 3, 4, -1, 7, 4, -3, 1:

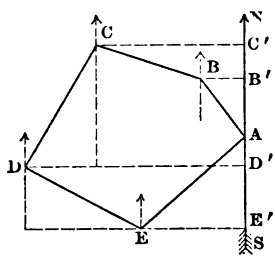
x,	y	(a)		$x_1 + x_2$,	$y_2 - y_1$	(b)	$y_1 + y_2$,	$x_2 - x_1$	(c)
1	3	−1	−12	5	-4	-20	2	3	6
4	-1	16	+ 7	11	5	55	3	3	9
7	4	7	+ 12	4	-3	-12	5	-10	-50
-3	1	−9	− 1	-2	2	-4	4	4	16
1	3								
		½(42 − 23)				½(55 − 36)			−½(31 − 50)

and the area, by either process, is 9½.

SURVEYING.

In taking bearings the north-and-south line is the initial line. The *latitude* of a point is its distance north or south of a given point; and the latitude of a line is the product of its length by the cosine of its bearing, *i.e.*, the length of its projection on the north line; the *departure* of a line is the product of its length by the sine of its bearing, *i.e.*, the length of its projection on the east line. Northings and eastings are positive; southings and westings are negative. The *meridian distance* of a point is its distance east or west from the origin, and the *double meridian distance* of a line is the sum of the meridian distances of its ends.

A *surveyor's chain* is four rods long and it is divided into a hundred *links*. Ten square chains make an acre.



Through any vertex, A, of a polygon ABCDE draw a north-and-south line NS;

through the other vertices B, C, D, E draw north-and-south lines, and east-and-west lines meeting NS in B′, C′, D′, E′;

then area pol ABCDE = − area triangle AB′B − area trap B′C′CB + area trap C′CDD′ + area trap D′DEE′ − area triangle EE′A

$$= \tfrac{1}{2}\left[AB' \cdot B'B + B'C' \cdot (B'B + C'C) + C'D' \cdot (C'C + D'D) + D'E' \cdot (D'D + E'E) + E'A \cdot E'E\right].$$

E.g., if a surveyor, starting from A, run N. 70° 20′ E. 6.37 chs. to B, thence N. 10° 15′ E. 4.28 chs. to C, thence S. 55° 35′ E. 12.36 chs. to D, thence S. 18° 45′ W. 14.96 chs. to E, thence N. 40° 55′

W. 11.15 chs. to F, thence N. 37° 15′ W. 8 chs. to A, then the area is 16.865 acres; and the work in tabular form is as follows:

	Bearing.	Dist. in Chains.	Dep.	M.D.	D.M.D.	Lat.	Double Area. +	Double Area. −
AB	N. 70° 20′ E.	6.37	5.998	5.998	5.998	2.144	12.860	
BC	N. 10° 15′ E.	4.28	.761	6.759	12.757	4.212	53.732	
CD	S. 55° 35′ E.	12.36	10.196	16.955	23.714	−6.985		165.642
DE	S. 18° 45′ W.	14.96	−4.809	12.146	29.101	−14.166		412.245
EF	N. 40° 55′ W.	11.15	−7.303	4.843	16.989	8.426	143.149	
FA	N. 37° 15′ W.	8.00	−4.843	0	4.843	6.369	30.845	

240.586, −577.887
240.586
−337.301

337.301 : 2 = 168.651 sq. chs. = 16.865 acres.

Column M.D. is got from column Dep. thus: $5.998 = 5.998$, $.761 + 5.998 = 6.759$, ⋯, $16.955 - 4.809 = 12.146$, ⋯;

column D.M.D. is got from column M.D. thus: $5.998 = 5.998$, $5.998 + 6.759 = 12.757$, $6.759 + 16.955 = 23.714$, ⋯;

the products of the double meridian distances by the latitudes give the double areas.

If in going about a field a surveyor turn continually to the left, counter-clock-wise, the area, computed as above, is positive; if to the right, clock-wise, it is negative.

EXAMPLES.

1. Find the area of the triangles, the rectangular coordinates of whose vertices are:

$$1, 3, \quad 2, 5, \quad 4, 1; \qquad 0, 0, \quad {}^{-}1, 2, \quad 2, 1.$$

2. So of the parallelogram whose vertices are:

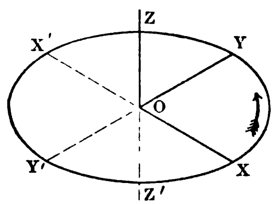
$${}^{+}3, {}^{+}5, \quad {}^{+}2, {}^{-}4, \quad {}^{-}3, {}^{-}5, \quad {}^{-}2, {}^{+}4.$$

3. If a surveyor, starting from A, run N. 22° 37′ E. 3.37 chs. to B, thence N. 80° 24′ E. 3.81 chs. to C, thence S. 41° 12′ E. 5.29 chs. to D, thence S. 62° 45′ W. $6.22\frac{1}{2}$ chs. to E; find the latitude and meridian distance of B, C, D, E, from A; find the bearing and distance of A from E; find the area.

*§ 11. PROJECTION OF A POLYGON ON A PLANE.

If two straight lines be at right angles, and if, while an observer looks along one of them toward the vertex, the other line revolve about the first as an axis, counter-clock-wise, then the revolving line generates a plane [geom.] whose *face* lies toward the observer; and the line of sight, directed from the plane to the eye, is the *normal to the plane.* Every point upon the revolving line generates a circle of the plane, with the vertex as centre; and arcs of these circles are $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right.$ when they reach $\left\{\begin{matrix}\text{in}\\ \text{against}\end{matrix}\right.$ the direction of their circles.

E.g., let XOZ be a right angle and let OX revolve about OZ in the direction XY;



then XOY is the plane generated, with its face toward Z, and OZ is the normal.

The *angle between two intersecting planes* whose faces are known is that one of their four diedrals which is generated by revolving one of the planes about their common section till its face coincides with the face of the other; *i.e.*, so that the normals of the two planes, at coincident points, shall coincide.

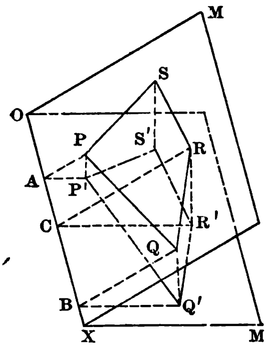
THEOR. 6. *If a polygon be projected from one plane to another, the area of the projection is the product of the area of the polygon by the cosine of the angle between the two planes.*

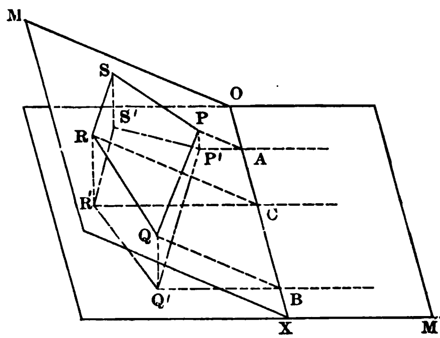
For, let PQRS be any polygon lying in the plane OM, and P'Q'R'S' its projection in the plane OM';

let the line of section OX be the x-axis in either plane;

let AP, BQ, ⋯ be the ordinates of P, Q, ⋯, and AP′, BQ′, ⋯ be the corresponding ordinates of P′, Q′, ⋯ ; [geom.

and let a be the angle between the planes;





then $\because$ $AP' = AP \cos a$, $BQ' = BQ \cos a, \cdots,$

$$\therefore\ AP'Q'B = -\tfrac{1}{2} AB\,(AP' + BQ')$$

$$= -\tfrac{1}{2} AB\,(AP + BQ) \cos a$$

$$= APQB \cos a.$$

So $BQ'R'C = BQRC \cos a$, and so on;

$\therefore$ $P'Q'R'S' = PQRS \cos a$. Q. E. D. [pr. 10 (c)

EXAMPLES.

1. When is the projection of a square: a square, a rectangle, a rhombus?

2. The sides of a rectangle are 3, 5; show how to project it into a square.

3. If the sun's altitude be 60°, find the area of the shadow on a horizontal plane, of the circle of unit radius that is: horizontal, vertical, perpendicular to the sun's rays, inclined to the sun's rays at an angle of 50° and with its horizontal diameter perpendicular to his rays (two cases).

IV. SPHERICAL TRIANGLES.

§ 1. GEOMETRIC PRINCIPLES.

If about the vertex of a triedral angle as centre a sphere be described, the *traces* of the three faces of the triedral upon the surface of the sphere are arcs of great circles, and together they form a *spherical triangle*, whose sides measure the face-angles, and whose angles measure the diedral angles, of the triedral. Since the sides are but measures of angles they may be expressed in angular measure, *e.g.*, a quadrant is an arc of one right angle or a right arc.

Two parts are of the *same species* when both are acute, or both obtuse, or both right.

THE LIMITED TRIANGLE.

It has been shown in geometry that in triangles whose parts are all positive and less than two right angles:

The sum of the three sides lies between naught and four right angles.

The sum of the three angles lies between two right angles and six right angles.

Each side is less than the sum of the other two.

Each angle is greater than the difference between two right angles and the sum of the other two.

Of any two unequal sides, the greater lies opposite the greater angle; and conversely.

Each side or angle is the supplement of the corresponding angle or side of the polar triangle.

If two sides of a triangle be equal, so are the opposite angles; and conversely.

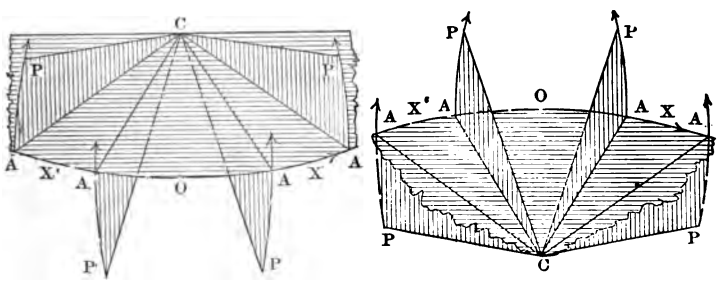
The perpendicular from the vertex of an isosceles triangle to the base bisects both the vertical angle and the base.

The area of a spherical triangle is to the area of the hemisphere as the excess of the sum of the three angles over two right angles is to four right angles.

THE GENERAL TRIANGLE.

In general the same conventions are used in the matter of *directed arcs* (great circles), positive and negative segments, and angles, and of general triangles upon the surface of the sphere, as in that of directed lines, positive and negative segments, and angles, and of general triangles in a plane.

The face of a sphere is the side from which rotation appears counter-clock-wise, and it may be either the convex or the concave surface; the reader is supposed to look always at the face of the sphere.



If the observer, so facing, look along an arc in its direction, then an *arc-normal* is an arc that crosses the first arc at right angles, from right to left, *i.e.*, in the direction of positive revolution of the positive end of the arc at the foot of the normal. Segments of any arc-normal reaching from the initial arc $\begin{cases}\text{in}\\ \text{against}\end{cases}$ the direction of the arc-normal are $\begin{cases}\text{positive}\\ \text{negative}\end{cases}$ *arc-ordinates.*

In the figure above, OA is the arc-abscissa and AP the arc-ordinate of the point P, as to the arc OX and origin O.

The plane of a directed arc faces toward that side from which the direction of the arc appears counter-clock-wise.

LEM. *That hemisphere whose arc-ordinates are positive, faces the same way as the plane of the initial arc.*

A diameter of a sphere that is perpendicular to a circle of the sphere is an *axis* of the circle, and the ends of the axis are the *poles* of the circle. The *positive pole* of a directed arc (great circle) is that pole whose arc-ordinate is positive; and an arc is the *polar* of its positive pole.

The angle between two arcs is the plane angle between the tangents at the vertex having the same directions as the arcs.

THEOR. 1. *The poles of two arcs lie on the polar of their point of intersection.* [geom.

The *polar* of a triangle ABC is that triangle whose sides l', m', n', are the polars of the vertices A, B, C, and whose vertices A$'$, B$'$, C$'$, are the positive poles of the sides l, m, n, of ABC.

COR. *If* A$'$B$'$C$'$ *be the polar of* ABC, *then reciprocally* ABC *is the polar of* A$'$B$'$C$'$.

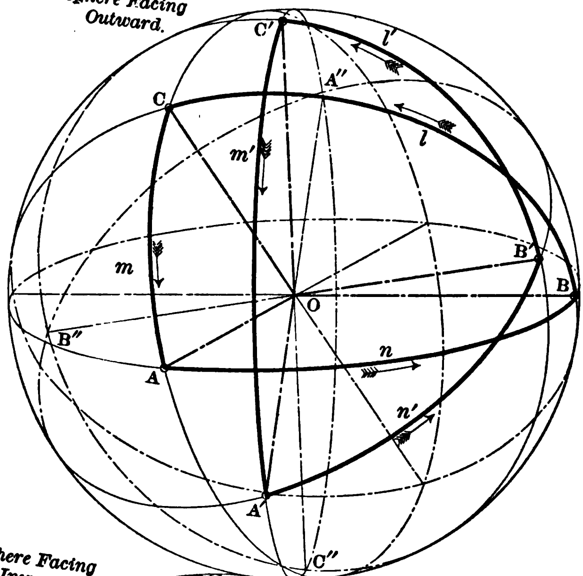
For $\quad l, m, n$ are the polars of A$'$, B$'$, C$'$, [df. polars
and $\quad$ A, B, C are the positive poles of l', m', n. Q. E. D.

THEOR. 2. *The* $\left\{\begin{matrix}\textit{sides}\\ \textit{angles}\end{matrix}\right.$ *of a spherical triangle and the corresponding* $\left\{\begin{matrix}\textit{angles}\\ \textit{sides}\end{matrix}\right.$ *of the polar triangle are supplementary.*

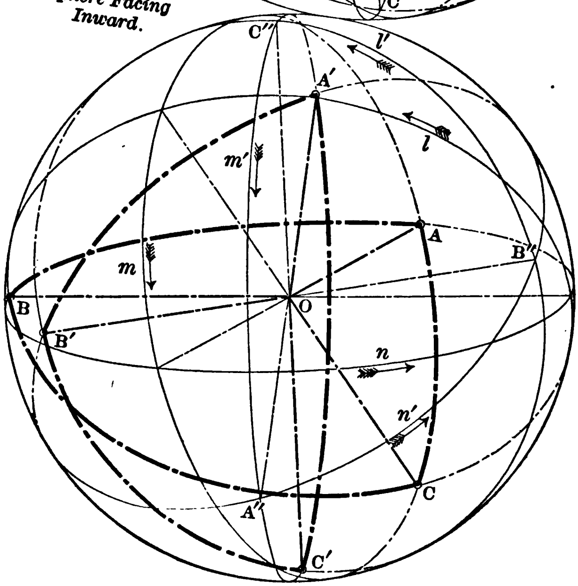
For, let the arc m turn about A till it coincides with n;
then $\because$ the positive pole B$'$ of the arc m turns from B$'$ to C$'$ on the polar of A,
and $\because$ positive rotation of an arc causes positive rotation of its positive pole on the polar of the vertex, [lm.
$\therefore$ arc B$'$C$'$ $= \angle mn =$ sup. $\angle$ A. Q. E. D.

NOTE. This theorem shows that, corresponding to any formula of the spherical triangle, there is a formula, found from the first by replacing A, B, C, a, b, c, by $\pi - a$, $\pi - b$, $\pi - c$, $\pi -$A, $\pi -$ B, $\pi -$ C; for these supplements are the parts A$'$, B$'$,

Sphere Facing Outward.



Sphere Facing Inward.



C$'$, a', b', c', of a spherical triangle, and so must satisfy the first formula. The second formula, in the same way, gives the first formula, and each of them is the *polar* of the other.

Note the three uses of the word *polar:* a circle may be the polar of a point, a triangle may be the polar of another triangle, and a formula may be the polar of another formula.

§ 2. PROPERTIES OF SPHERICAL ANGLES.

THEOR. 3. *If upon the surface of a sphere whose centre is* C, XOL *be any spherical angle, if* OX$'$, OL$'$ *be tangents to* OX, OL *at* O, *and reach in the same directions, if* AK *be any arc-normal to* OX *and if* A$'$K$'$ *be perpendicular to the plane* COX *at* A$'$, *the intersection of* CA, OX$'$, *and reach from the plane on the same side as* AK: *then are* OX$'$, OL$'$ *normal in the planes* COX, COL *to* CO; *and* A$'$K$'$ *is normal in the planes* X$'$OL$'$, CAK *to* OX$'$, CA.

For ∵ OX$'$, OL$'$ are perpendiculars to CO in the planes COX, COL, and reach in the direction of revolution for these planes, [constr.

∴ they are normals to CO in these planes; [df. nor.

and ∵ A$'$K$'$ is perpendicular to OX$'$, CA in the planes X$'$OL$'$, CAK, and reaches in the direction AK,

and AK is the direction of positive revolution of CA in the plane CAK, and of positive revolution of OX on the spherical surface, *i.e.*, of OX$'$ in the plane X$'$OL$'$,

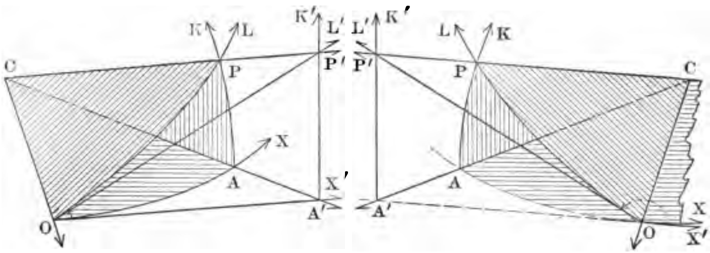
∴ A$'$K$'$ is normal to CA, OX$'$ in the planes CAK, X$'$OL$'$. [Q. E. D.

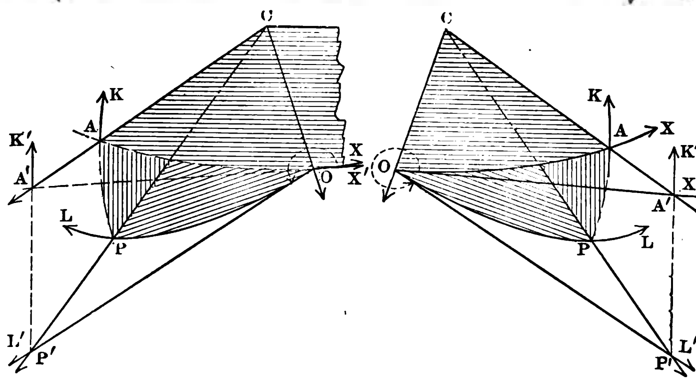
COR. *If* CPP$'$ *be the intersection of planes* COL, CAK, *and* P$'$ *be on* OL$'$, *then* OP$'$ *is the distance of* P$'$ *as to* OX$'$ *in plane* X$'$OL$'$, *and the ordinate of* P$'$ *as to* CO *in plane* COL; *and for both uses* OP$'$ *is a segment of the same line,* OL$'$, *and has the same sign.*

So OA$'$ *is the abscissa of* P$'$ *as to* OX$'$ *in plane* X$'$OL$'$, *and the ordinate of* A$'$ *as to* CO *in plane* COX; *and for both uses* OA$'$ *is a segment of the same line,* OX$'$, *and has the same sign.*

So A$'$P$'$ *is the ordinate of* P$'$ *as to* OX$'$ *in plane* X$'$OL$'$ *and the*

ordinate of P$'$ *as to* CA *in plane* CAK; *and for both uses* A$'$P$'$ *is a segment of the same line,* A$'$K$'$, *and has the same sign.*





THEOR. 4. *If r, x, y, be the arc-distance, arc-abscissa, and arc-ordinate, as to the initial arc, of any point on the terminal arc of a spherical angle* O; *then:*

$$\sin \mathrm{O} = \sin y : \sin r, \quad \cos \mathrm{O} = \tan x : \tan r,$$
$$\tan \mathrm{O} = \tan y : \sin x, \quad \cos r = \cos x \cdot \cos y.$$

For, let O be any spherical angle and let $r, x, y \equiv$ OP, OA, AP;

then $$\sin \mathrm{O} = \mathrm{A'P'} : \mathrm{OP'} = \frac{\mathrm{A'P'}}{\mathrm{CP'}} : \frac{\mathrm{OP'}}{\mathrm{CP'}} = \sin y : \sin r.$$

$$\cos \mathrm{O} = \mathrm{OA'} : \mathrm{OP'} = \frac{\mathrm{OA'}}{\mathrm{CO}} : \frac{\mathrm{OP'}}{\mathrm{CO}} = \tan x : \tan r.$$

$$\tan \mathrm{O} = \mathrm{A'P'} : \mathrm{OA'} = \frac{\mathrm{A'P'}}{\mathrm{CA'}} : \frac{\mathrm{OA'}}{\mathrm{CA'}} = \tan y : \sin x.$$

$$\cos r = \mathrm{CO} : \mathrm{CP'} = \frac{\mathrm{CO}}{\mathrm{CA'}} \cdot \frac{\mathrm{CA'}}{\mathrm{CP'}} = \cos x \cdot \cos y. \qquad \text{Q. E. D.}$$

§ 3. RIGHT TRIANGLES.

THEOR. 5. *If* PQR *be a spherical triangle, right-angled at* R, *then:*

$$\sin p = \sin r \ \sin P, \qquad \sin q = \sin r \ \sin Q,$$
$$\cos P = \tan q \ \cot r, \qquad \cos Q = \tan p \ \cot r,$$
$$\sin q = \tan p \ \cot P, \qquad \sin p = \tan q \ \cot Q,$$
$$\cos r = \cos p \ \cos q, \qquad \cos r = \cot P \ \cot Q,$$
$$\cos P = \cos p \ \sin Q, \qquad \cos Q = \cos q \ \sin P.$$

For ∵ in the triangle PQR, arcs p, q, r are the arc-ordinate, arc-abscissa, arc-distance of $\angle$ P,

and in the symmetric triangle QPR, as seen through the paper from the back, arcs q, p, r are the arc-ordinate, arc-abscissa, arc-distance of $\angle$ Q,

∴ $\sin P = \sin p : \sin r$, $\sin Q = \sin q : \sin r$, [th. 4

∴ $\sin p = \sin r \cdot \sin P$, $\sin q = \sin r \cdot \sin Q$. Q.E.D.

So ∵ $\cos P = \tan q : \tan r$, $\cos Q = \tan p : \tan r$,

∴ $\cos P = \tan q \cdot \cot r$, $\cos Q = \tan p \cdot \cot r$. Q.E.D.

So ∵ $\tan P = \tan p : \sin q$, $\tan Q = \tan q : \sin p$,

∴ $\sin q = \tan p \cdot \cot P$, $\sin p = \tan q \cdot \cot Q$. Q.E.D.

So $\cos r = \cos p \cdot \cos q$ Q.E.D.

$= \sin p \cot p \cdot \sin q \cot q = \sin q \cot p \cdot \sin p \cot q$

$= \cot P \cdot \cot Q$. Q.E.D.

So $\cos P = \tan q \cdot \cot r = \dfrac{\cos r}{\cos q} \cdot \dfrac{\sin q}{\sin r} = \cos p \cdot \sin Q$. Q.E.D.

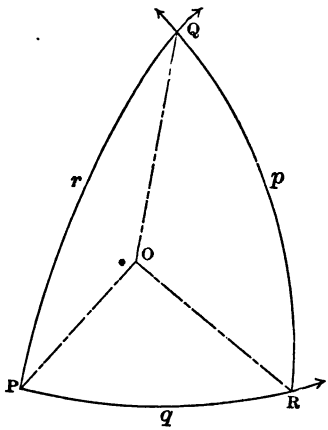
So $\cos Q = \cos q \cdot \sin P$. Q.E.D.

NAPIER'S RULES: By an ingenious device of Lord Napier these ten formulæ are remembered by two simple rules:

Ignore the right angle; take the two sides, and replace the hypothenuse and two oblique angles by their complements; and of the five parts so found call any one the middle part, *the two lying next it* adjacent parts, *and the two others the* opposite parts, *then:*

sin m*id*. part = prod. *tan* *adja*. parts,

sin m*id*. part = prod. cos *oppo*. parts.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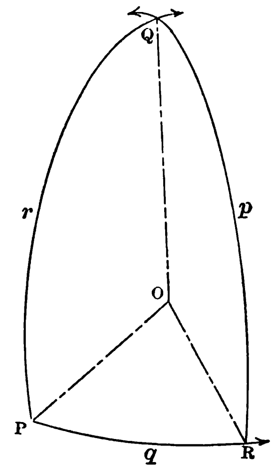
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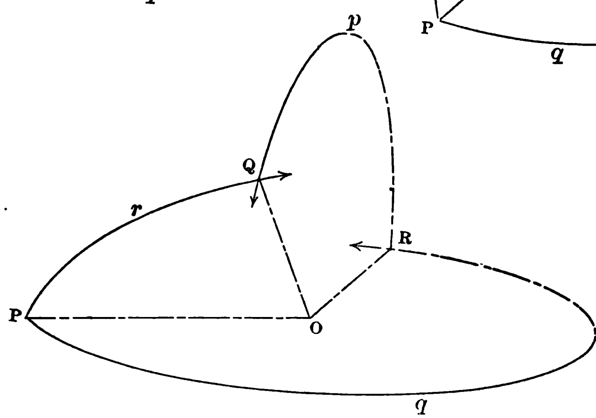
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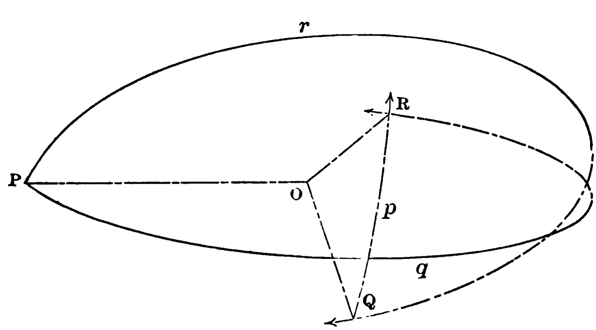
r
R
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p
q
Q

Cor. 1. *If in a right spherical triangle another part beside the right angle be right, the triangle is biquadrantal.*

For, $\because$ $\cos r = \cos p \cdot \cos q$,

and $\cos \text{P} = \cos p \sin \text{Q}$, and $\sin \text{Q} \neq 0$, [th. 5.

$\cos \text{Q} = \cos q \sin \text{P}$, and $\sin \text{P} \neq 0$,

$\therefore$ if p be right, so are r, P,

if P be right, so are r, p,

and if r be right, so are p, P or q, Q. Q. E. D.

Cor. 2. *In a right spherical triangle, not biquadrantal, the hypothenuse is nearer right than either oblique side.*

For $\because$ $\cos r = \cos p \cos q$, and $\cos p, \cos q < 1$,

$\therefore$ $\cos r < \cos p, \cos q$, [th. 5

$\therefore$ r is nearer right than either p or q. Q. E. D.

Cor. 3. *In a right spherical triangle, not biquadrantal, an oblique angle is nearer right than its opposite side.*

For $\because$ $\cos \text{P} = \cos p \sin \text{Q}$, and $\sin \text{Q} < 1$, [th. 5

$\therefore$ $\cos \text{P} < \cos p$,

$\therefore$ P is nearer right than p. Q. E. D.

Cor. 4. *In a right spherical triangle, whose parts are all positive and less than two right angles, an oblique angle and its opposite side are of the same species.*

For $\because$ $\cos \text{P} = \cos p \sin \text{Q}$,

and $\sin \text{Q}$ is positive and not 0,

$\therefore$ $\cos \text{P}$, $\cos p$ are both positive, both negative, or both zero;

$\therefore$ P, p are both acute, both obtuse, or both right. Q. E. D.

Cor. 5. *In a right spherical triangle whose parts are all positive and less than two right angles, if the hypothenuse be* $\left\{\begin{array}{l}\textit{acute,}\\ \textit{obtuse,}\end{array}\right.$ *the other two sides are of* $\left\{\begin{array}{l}\textit{the same}\\ \textit{opposite}\end{array}\right.$ *species, and so are the two oblique angles; and conversely.*

For $\because \cos r = \cos p \cos q$,

$\therefore$ if r be $\begin{cases}\text{acute,}\\ \text{obtuse,}\end{cases}$ then $\cos r$ is $\begin{cases}\text{positive,}\\ \text{negative,}\end{cases}$

$\therefore \cos p, \cos q$ are $\begin{cases}\text{both positive or both negative,}\\ \text{one positive, the other negative,}\end{cases}$

$\therefore p, q$ are $\begin{cases}\text{both acute or both obtuse.}\\ \text{one acute, the other obtuse.}\end{cases}$

So for P, Q; prove from the equation $\cos r = \cot P \cot Q$.

EXAMPLES.

1. $\sin^2 \frac{1}{2} r = \sin^2 \frac{1}{2} p \cos^2 \frac{1}{2} q + \cos^2 \frac{1}{2} p \sin^2 \frac{1}{2} q$.
2. $\tan^2 \frac{1}{2} p = \tan \frac{1}{2}(r+q) \tan \frac{1}{2}(r-q)$.
3. $\tan^2 \frac{1}{2} P = \sin(r-q) : \sin(r+q)$.
4. $\tan \frac{1}{2} P = \sin(r-q) : \sin p \cos q = \sin p \cos q : \sin(r+q)$.
5. $\sin(p-q) = \sin p \tan \frac{1}{2} P - \sin q \tan \frac{1}{2} Q$.
6. $\tan^2 \frac{1}{2} p = \tan \frac{1}{2}(Q + P - 90°) : \tan \frac{1}{2}(Q - P + 90°)$.
7. $\tan^2 \frac{1}{2} r = -\cos(P+Q) : \cos(P-Q)$.

§ 4. SOLUTION OF RIGHT SPHERICAL TRIANGLES.

PROB. 1. TO SOLVE A RIGHT SPHERICAL TRIANGLE.

Take each of the two given parts in turn for middle part, and apply that formula which brings in the other given part.

Take the remaining part for middle part, and apply that formula which brings in both of the parts just found.

Solve the equations so found for the parts sought.

CHECK: *make the part last found the middle part, and apply that formula which brings in both the given parts.*

The check is defective in this, that it tests the functions, but not the angles got from these functions, *i.e.*, not the final results; more perfect checks are got from the general formulæ.

The check is applied to the sine of the part last found; if the two values got for this sine, natural or logarithmic, differ

by not more than three units in the last decimal place, the work is probably right, since the defects of the tables permit this discrepancy in the two results; if such discrepancy exist, the mean of the two values may be used.

THE PARTS ALL POSITIVE AND LESS THAN TWO RIGHT ANGLES.

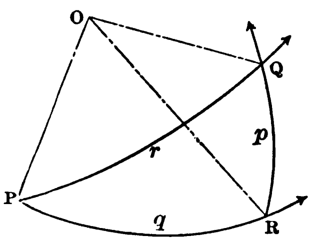
(*a*) *Given* p, q, *the two sides about the right angle:*

then $\sin q = \tan p \cot \mathrm{P}$, $\therefore\ \cot \mathrm{P} = \cot p \sin q$;

$\sin p = \tan q \cot \mathrm{Q}$, $\therefore\ \cot \mathrm{Q} = \sin p \cot q$;

$\cos r = \cot \mathrm{P} \cot \mathrm{Q}$; *check* $\cos r = \cos p \cos q$.

One triangle is always possible, and but one; the parts r, P, Q are determined without ambiguity by the formulæ.



(*b*) *Given* r, p, *the hypothenuse and one side:*

then $\cos r = \cos p \cos q$, $\therefore\ \cos q = \cos r : \cos p$;

$\sin p = \sin r \sin \mathrm{P}$, $\therefore\ \sin \mathrm{P} = \sin p : \sin r$;

$\cos \mathrm{Q} = \cos q \sin \mathrm{P}$; *check* $\cos \mathrm{Q} = \tan p \cot r$.

A triangle is possible only when r is nearer right than p, or when r, p are both right. [th. 5 cr. 2

The formula gives two values to P, supplementary, but only that value of P is admissible which is of the same species as p.

The parts q, P, Q are determined without ambiguity by the formulæ, unless p, r be both right, when $\cos \mathrm{Q}$, $\cos q$ are indeterminate; then $q = \mathrm{Q}$, and P is right.

(*c*) *Given* P, q, *an oblique angle and the adjacent side:*

then $\sin q = \tan p \cot \mathrm{P}$, $\therefore\ \tan p = \sin q \tan \mathrm{P}$;

$\cos \mathrm{P} = \tan q \cot r$, $\therefore\ \cot r = \cot q \cos \mathrm{P}$;

$\cos \mathrm{Q} = \tan p \cot r$; *check* $\cos \mathrm{Q} = \cos q \sin \mathrm{P}$.

One triangle is always possible, and but one; the parts p, r, Q are determined without ambiguity by the formulæ.

(*d*) *Given* P, p, *an oblique angle and the opposite side:*

then $\quad \sin p = \sin r \sin \text{P}, \quad \therefore \sin r = \sin p : \sin \text{P}$;

$\quad \cos \text{P} = \cos p \sin \text{Q}, \quad \therefore \sin \text{Q} = \cos \text{P} : \cos p$;

$\quad \sin q = \sin r \sin \text{Q}$; *check* $\sin q = \tan p \cot \text{P}$.

If p, P be not of the same species, no triangle is possible, [th. 5 cr. 4

nor if p be nearer right than P. [th. 5 cr. 3

If p, P be equal, but not right,

then $\because$ $\sin r$, $\sin \text{Q}$, $\sin q$ are all 1,

$\therefore$ r, Q, q are all right,

and the triangle is biquadrantal.

If p, P be both right,

then r is right, the triangle is biquadrantal, [geom.

and Q, q are indeterminate.

If P be nearer right than p, and of the same species,

then two triangles are always possible.

For $\because$ r, Q, q are all found from their sines,

and to every sine correspond two angles, supplementary,

$\therefore$ r, Q, q, may each have two values;

but $\because$ Q, q are of the same species, [th. 5 cr. 4

$\therefore$ with r acute, q, Q are of the same species with p, P, and there is but one value for each of them; [th. 5 cr. 5

so, with r obtuse, q, Q are of the opposite species to p, P, and there is but one value for each of them;

$\therefore$ two triangles, and but two, are formed with the data:

viz.: that wherein r is acute, and q, Q are of the same species with p, P;

and that wherein r is obtuse, and q, Q are of the opposite species to p, P.

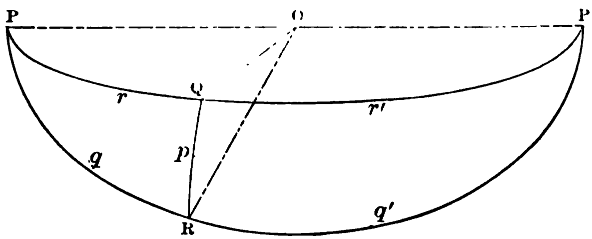
This ambiguity appears directly from the figure.

Produce the arcs PQ, PR to meet at P′;

then $\because$ P = P′, [geom.

and $\measuredangle$s PRQ, P′RQ are both right, [hyp.

$\therefore$ two right triangles exist, PQR, P′QR, that have the same two parts given, p, P, and the remaining parts of the one triangle supplementary to those of the other.



(*e*) *Given* r, P, *the hypothenuse and an oblique angle:*

then $\cos P = \tan q \cot r$, $\quad\therefore\ \tan q = \tan r \cos P$;

$\cos r = \cot P \cot Q$, $\quad\therefore\ \cot Q = \cos r \tan P$;

$\sin p = \tan q \cot Q$; *check* $\sin p = \sin r \sin P$.

One triangle is always possible, and but one, for the part p is of the same species with P, and so can have but one of two possible values; the parts p, q, Q are determined without ambiguity by the formulæ.

(*f*) *Given* P, Q, *the two oblique angles:*

then $\cos P = \cos p \sin Q$, $\quad\therefore\ \cos p = \cos P : \sin Q$;

$\cos Q = \cos q \sin P$, $\quad\therefore\ \cos q = \cos Q : \sin P$;

$\cos r = \cos p \cos q$; *check* $\cos r = \cot P \cot Q$.

The parts p, q, r are determined without ambiguity by the formulæ; but the solution is possible only when

$\cos P : \sin Q$, $\quad \cos Q : \sin P$, $\quad$ each < 1;

i.e., when P is nearer right than co-Q,

and when Q is nearer right than co-P.

E.g., given $p = 72°$, $q = 125°$, $\text{R} = 90°$:

$$\begin{array}{ll}
\cot \text{P} = \cot p \sin q, & \cot \text{Q} = \sin p \,\overline{\cot}\, q, \\
\quad 9.511776 & \quad 9.978206 \\
\quad \underline{9.913365} & \quad \underline{9.845227} \\
\quad 9.425141 & \quad 9.823433 \text{ neg.} \\
\text{P} = 75° \, 5' \, 45''. & \text{Q} = 123° \, 39' 40''. \quad [\text{th. 5 cr. 4}
\end{array}$$

$$\begin{array}{ll}
\cos r = \cot \text{P} \,\overline{\cot}\, \text{Q}, & \textit{check } \cos r = \cos p \,\overline{\cos}\, q. \\
\quad 9.425141 & \quad 9.489982 \\
\quad \underline{9.823433} & \quad \underline{9.758591} \\
\quad 9.248574 \text{ neg.} & \quad 9.248573 \text{ neg.}
\end{array}$$

$r = 100° \, 12' 34''$. [th. 5 cr. 5.

So, given $r = 60°$, $p = 135°$, $\text{R} = 90°$:

$$\begin{array}{ll}
\cos q = \cos r : \overline{\cos}\, p, & \sin \text{P} = \sin p : \sin r, \\
\quad 9.698970 & \quad 9.849485 \\
\quad \underline{9.849485} & \quad \underline{9.937531} \\
\quad 9.849485 \text{ neg.} & \quad 9.911954 \\
q = 135°. \quad [\text{th. 5 cr. 5.} & \text{P} = 125° \, 15' 51''. \quad [\text{th. 5 cr. 4}
\end{array}$$

$$\begin{array}{ll}
\cos \text{Q} = \overline{\cos}\, q \sin \text{P}, & \textit{check } \cos \text{Q} = \overline{\tan}\, p \cot r. \\
\quad 9.849485 & \quad 0.000000 \\
\quad \underline{9.911954} & \quad \underline{9.761439} \\
\quad 9.761439 \text{ neg.} & \quad 9.761439 \text{ neg.}
\end{array}$$

$\text{Q} = 125° \, 15' \, 52''$. [th. 5 cr. 4.

So, given $\text{P} = 125° \, 31'$, $p = 151°$, $\text{R} = 90°$:

$$\begin{array}{ll}
\sin r = \sin p : \sin \text{P}, & \sin \text{Q} = \overline{\cos}\, \text{P} : \overline{\cos}\, p, \\
\quad 9.685571 & \quad 9.764131 \\
\quad \underline{9.910596} & \quad \underline{9.941819} \\
\quad 9.774975 & \quad 9.822312 \\
r = 36° \, 33' 27'', & \text{Q} = 138° \, 22' 39'', \\
\text{or} \quad 143° \, 26' 33''. [\text{th. 5 cr. 5.}] & \text{or } 41° \, 37' \, 21''. \quad [\text{th. 5 cr. 5}
\end{array}$$

$$\begin{array}{ll}
\sin q = \sin r \sin \text{Q}, & \textit{check } \sin q = \overline{\tan}\, p \,\overline{\cot}\, \text{P}. \\
\quad 9.774975 & \quad 9.743752 \\
\quad \underline{9.822312} & \quad \underline{9.853535} \\
\quad 9.597287 & \quad 9.597287
\end{array}$$

$q = 156° \, 41' 41''$, or $23° \, 18' \, 19''$. [th. 5 cr. 4.

So, given $P = 119° 47'$, $Q = 55° 9'$, $R = 90°$:

$\cos p = \overline{\cos} P : \sin Q$, $\cos q = \cos Q : \sin P$,

9.696113	9.756963
9.914158	9.938475
9.781955 neg.	9.818488

$p = 127° 14' 56''$. $q = 48° 49' 20''$. [th. 5 cr. 4

$\cos r = \overline{\cos} p \cos q$, *check* $\cos r = \overline{\cot} P \cot Q$.

9.781955	9.757638
9.818488	9.842805
9.600443 neg.	9.600443 neg.

$r = 113° 29' 7''$. [th. 5 cr. 5.

With angles near the ends of a quarter, use the special table for small angles, or the formulæ given in exs. 1–7, § 3.

QUADRANTAL TRIANGLES. *Find the polar of the given triangle; it is a right triangle; solve it and take the supplements of the parts thus found for the parts sought in the given triangle.*

A biquadrantal triangle is indeterminate unless either the base or the vertical angle be given.

ISOSCELES TRIANGLES. *Draw an arc from the vertex to the middle of the base, thereby dividing the given triangle into two equal right triangles; solve one of these triangles.*

If only the base and the vertical angle be given, there are two triangles, one triangle, or none, according as the base is less than, equal to, or greater than the vertical angle; if only the two equal sides or the two equal angles be given, there is an infinite number of triangles; otherwise, subject to the conditions just found (b, f), there is one triangle, and but one.

OBLIQUE TRIANGLES. In most cases a perpendicular may fall from a vertex of an oblique triangle to the opposite side in such manner that one of the right triangles thus formed shall contain two of the three known parts, and the other right triangle shall contain one of them.

Solve these right triangles in order and so combine the parts as to find the parts sought in the given triangle.

EXAMPLES.

1...10. Solve the right triangles, and check the work, given:

1. $p = 116°$, $q = 16°$, $\text{R} = 90°$.
[97° 39′ 24″, 17° 41′ 40″, 114° 55′ 20″

2. $r = 140°$, $p = 20°$, $\text{R} = 90°$.
[32° 8′ 48″, 115° 42′ 24″, 144° 36′ 29″

3. $\text{P} = 80° 10'$, $q = 155° 46'$, $\text{R} = 90°$.
[67° 6′ 23″, 153° 57′ 34″, 110° 46′ 40″

4. $\text{P} = 100°$, $p = 112°$, $\text{R} = 90°$.
[27° 36′ 59″, 109° 41′ 49″, 25° 52′ 33″,
or 152° 23′ 1″, 70° 18′ 11″, 154° 7′ 27″

5. $r = 120°$, $\text{P} = 120°$, $\text{R} = 90°$.
[131° 24′ 34″, 40° 53′ 36″, 49° 6′ 24″

6. $\text{P} = 60° 47'$, $\text{Q} = 57° 16'$, $\text{R} = 90°$.
[54° 31′ 52″, 51° 43′ 1″, 68° 55′ 50″

7. $r = 140°$, $p = 140°$, $\text{R} = 90°$.

8. $r = 120°$, $\text{P} = 90°$, $\text{R} = 90°$.

9. $\text{A} = 80°$, $a = 90°$, $b = 37°$.
[97° 27′ 18″, 102° 27′ 1″, 36° 20′ 49″

10. $\text{B} = 50°$, $b = 130°$, $c = 90°$.

11...13. Solve the isosceles triangles, given:

11. $a = 70°$, $b = 70°$, $\text{A} = 30°$. [157° 39′ 34″, 134° 24′ 30″

12. $a = 30°$, $\text{A} = 70°$, $\text{B} = 70°$.

13. $a = 119°$, $b = 119°$, $c = 85°$. [113° 57′ 11″, 72° 26′ 22″

14. Show how to solve an oblique triangle by means of right triangles if the three sides be given.

With the figures as on page 102,

show that $\cos(c - q) : \cos q = \cos a : \cos b$, [th. 4

then that $\tan(\frac{1}{2}c - q) = \tan\frac{1}{2}(a + b) \cdot \tan\frac{1}{2}(a - b) : \tan\frac{1}{2}c$,

whence q is found; and the rest follows easily.

So if the three angles be given.

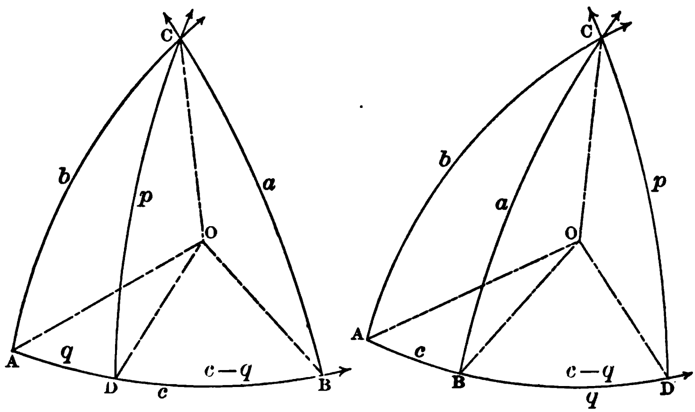
15. If PQR be a right spherical triangle, show that
$\text{P} \sim \text{Q} < 90° < \text{P} + \text{Q} < 270°$.

§ 5. GENERAL PROPERTIES OF SPHERICAL TRIANGLES.

LAW OF COSINES.

THEOR. 6. *In any spherical triangle* ABC,

$$\cos a = \cos b \cos c + \sin b \sin c \cos A, \cdots$$



For, draw DC arc-ordinate of C as to AB,
and let $p,\ q,\ c-q \equiv$ DC, AD, DB ;
then ∵ $\triangle$s ADC, BDC are right-angled at D,

and $\cos b = \cos p \cos q,\quad \cos a = \cos p \cos (c-q),$ [th. 4

∴ $\cos a = (\cos b : \cos q)(\cos c \cos q + \sin c \sin q)$ [ad. th.

$= \cos b \cos c + \sin c \cos b \tan q$;

and ∵ $\tan q = \cos A \tan b,$ [th. 4

∴ $\cos a = \cos b \cos c + \sin b \sin c \cos A$;

and so for $\cos b$, $\cos c$. Q. E. D.

COR. 1. *In any spherical triangle whose parts are all positive and less than two right angles, a side and its opposite angle are of the same species if there be another side as near right as the given side.*

For let the side c be as near right as the given side a;

then $\because$ $\cos c \not> \cos a$, $\cos b \not< 1$,

$\therefore$ $\cos b \cos c \not< \cos a$ or $\cos b \cos c = \cos a = 0$,

$\therefore$ $\cos a$, $\cos a - \cos b \cos c$, are positive, negative, or zero together;

but $\because$ $\cos A = (\cos a - \cos b \cos c) : \sin b \sin c$,

and $\because$ $\sin b \sin c$ is positive, [II. th. 3

$\therefore$ $\cos A$, $\cos a - \cos b \cos c$, are positive, negative, or zero together,

$\therefore$ $\cos A$, $\cos a$, are positive, negative, or zero together,

$\therefore$ A, a are both acute, both obtuse, or both right. Q. E. D.

COR. 2. $\sin \frac{1}{2} A = \sqrt{[\sin (s-b) \sin (s-c) : \sin b \sin c]}, \cdots.$
[$s = \frac{1}{2}(a+b+c)$

For $\because$ $2 \sin^2 \frac{1}{2} A = 1 - \cos A$, [II. th. 13 cr.

$$
\begin{aligned}
&= 1 - [(\cos a - \cos b \cos c) : \sin b \sin c] && \text{[th. 6}\\
&= [\cos b \cos c + \sin b \sin c - \cos a] : \sin b \sin c\\
&= [\cos (b-c) - \cos a] : \sin b \sin c && \text{[ad. th.}\\
&= -2 \sin \tfrac{1}{2} (b-c+a) \sin \tfrac{1}{2} (b-c-a) : \sin b \sin c\\
&= 2 \sin \tfrac{1}{2} (a+b-c) \sin \tfrac{1}{2} (a-b+c) : \sin b \sin c\\
&= 2 \sin (s-c) \sin (s-b) : \sin b \sin c,
\end{aligned}
$$

$\therefore$ $\sin \frac{1}{2} A = \sqrt{[\sin (s-b) \sin (s-c) : \sin b \sin c]}$;

and so for $\sin \frac{1}{2} B$, $\sin \frac{1}{2} C$ Q. E. D.

COR. 3. $\cos \frac{1}{2} A = \sqrt{[\sin s \sin (s-a) : \sin b \sin c]}.$

For $\because$ $2 \cos^2 \frac{1}{2} A = 1 + \cos A$ [II. th. 13 cr.

$$
\begin{aligned}
&= 1 + [(\cos a - \cos b \cos c) : \sin b \sin c] && \text{[th. 6}\\
&= (\cos a - \cos b \cos c + \sin b \sin c) : \sin b \sin c\\
&= [\cos a - \cos (b+c)] : \sin b \sin c && \text{[ad. th.}\\
&= -2 \sin \tfrac{1}{2} (a+b+c) \sin \tfrac{1}{2} (a-b-c) : \sin b \sin c\\
&= 2 \sin \tfrac{1}{2} (a+b+c) \sin \tfrac{1}{2} (-a+b+c) : \sin b \sin c\\
&= 2 \sin s \sin (s-a) : \sin b \sin c,
\end{aligned}
$$

$\therefore$ $\cos \frac{1}{2} A = \sqrt{[\sin s \sin (s-a) : \sin b \sin c]}$;

and so for $\cos \frac{1}{2} B$, $\cos \frac{1}{2} C$. Q. E. D.

COR. 4. $\tan\frac{1}{2}A = \sqrt{[\sin(s-b)\sin(s-c) : \sin s \sin(s-a)]}, \cdots.$

THEOR. 7. *In any spherical triangle* ABC,

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a, \cdots$$

For $\because$ in the polar triangle $A'B'C'$,

$$a' = \text{sup. } A, \cdots, \quad A' = \text{sup. } a, \cdots,$$

and $\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A'$, [th. 6

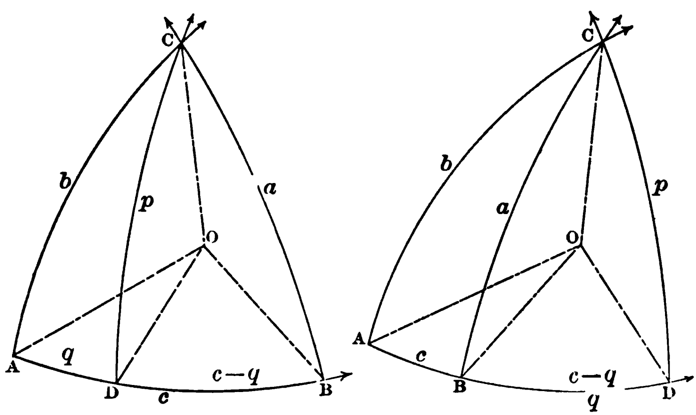
$\therefore$ $\cos \text{sup. } A = \cos \text{sup. } B \cos \text{sup. } C$
$+ \sin \text{sup. } B \sin \text{sup. } C \cos \text{sup. } a$,

$\therefore$ $\cos A = -\cos B \cos C + \sin B \sin C \cos a$. Q.E.D.

Another proof, analogous to that of th. 6, is this:

For, draw DC arc-ordinate of C as to AB,

and let p, Q, C + Q $\equiv$ DC, $\angle$ BCD, $\angle$ ACD;



then $\because$ $\cos A = \cos p \sin(C+Q)$, $\cos \text{sup. } B = \cos p \sin Q$, [th. 5

$\therefore$ $\cos A = -(\cos B : \sin Q)(\sin C \cos Q + \cos C \sin Q)$ [ad. th.

$= -\cos B \cos C - \cos B \sin C \cot Q$;

and $\because$ $\cot Q = -\cos a \tan B$, [th. 5

$\therefore$ $\cos A = -\cos B \cos C + \sin B \sin C \cos a$;

and so for $\cos B$, $\cos C$. Q.E.D.

COR. 1. *In any spherical triangle, whose parts are all positive and less than two right angles, a side and its opposite angle are of the same species if there be another angle as near right as the given angle.*

Prove as th. 6 cr. 1 was proved.

COR. 2. $\sin \frac{1}{2} a = \sqrt{[\sin \text{E} \sin (\text{A} - \text{E}) : \sin \text{B} \sin \text{C}]}, \cdots.$
$[\text{E} = \frac{1}{2}(\text{A} + \text{B} + \text{C} - \pi)$

Prove from the formula

$$\cos \text{A} = -\cos \text{B} \cos \text{C} + \sin \text{B} \sin \text{C} \cos a$$

as th. 3 cr. 2 was proved from the formula

$$\cos a = \cos b \cos c + \sin b \sin c \cos \text{A}.$$

COR. 3. $\cos \frac{1}{2} a = \sqrt{[\sin (\text{B} - \text{E}) \sin (\text{C} - \text{E}) : \sin \text{B} \sin \text{C}]}, \cdots.$

COR. 4. $\tan \frac{1}{2} a = \sqrt{[\sin \text{E} \sin (\text{A} - \text{E}) : \sin (\text{B} - \text{E}) \sin (\text{C} - \text{E})]}.$

NOTE. The formulæ of th. 7 and its corollaries are the polar formulæ of th. 6 and its corollaries, and may be proved by their aid if a be replaced by sup. A $\cdots$, A by sup. a, $\cdots$, s by sup. E, $s - a$ by A $-$ E, $\cdots$.

LAW OF SINES.

THEOR. 8. *In any spherical triangle* ABC

$$\sin a : \sin b : \sin c = \sin \text{A} : \sin \text{B} : \sin \text{C}.$$

For, draw DC arc-ordinate of C as to AB, and let $p \equiv \text{DC}$;
[fig. page 104

then $\because$ $\sin p = \sin b \sin \text{A}$, [th. 4

and $\sin p = \sin a \sin \text{sup. B} \doteq \sin a \sin \text{B}$,

$\therefore$ $\sin a \cdot \sin \text{B} = \sin b \cdot \sin \text{A}$,

$\therefore$ $\sin a : \sin b = \sin \text{A} : \sin \text{B}$.

So $\sin b : \sin c = \sin \text{B} : \sin \text{C}$.

$\therefore$ $\sin a : \sin b : \sin c = \sin \text{A} : \sin \text{B} : \sin \text{C}$. Q. E. D.

Translated into words, th. 8 is:

In any spherical triangle the sines of the sides are proportional to the sines of the opposite angles.

NAPIER'S ANALOGIES.

THEOR. 9. *In any spherical triangle* ABC:

$$\tan\tfrac{1}{2}(a+b)=\cos\tfrac{1}{2}(\text{A}-\text{B}):\cos\tfrac{1}{2}(\text{A}+\text{B})\cdot\tan\tfrac{1}{2}c,$$
$$\tan\tfrac{1}{2}(a-b)=\sin\tfrac{1}{2}(\text{A}-\text{B}):\sin\tfrac{1}{2}(\text{A}+\text{B})\cdot\tan\tfrac{1}{2}c.$$
$$\tan\tfrac{1}{2}(\text{A}+\text{B})=\cos\tfrac{1}{2}(a-b):\cos\tfrac{1}{2}(a+b)\cdot\cot\tfrac{1}{2}\text{C},$$
$$\tan\tfrac{1}{2}(\text{A}-\text{B})=\sin\tfrac{1}{2}(a-b):\sin\tfrac{1}{2}(a+b)\cdot\cot\tfrac{1}{2}\text{C},$$

For let $k\equiv\sin a:\sin\text{A}\equiv\sin b:\sin\text{B}\equiv\sin c:\sin\text{C}$
$=(\sin a\pm\sin b):(\sin\text{A}\pm\sin\text{B})$; [law of sines

then $\because$ $\cos a-\cos b\cos c=\sin b\sin c\cos\text{A}$ [law of cosines
$=k\sin c\cos\text{A}\sin\text{B}$,

and $\cos b-\cos a\cos c=\sin a\sin c\cos\text{B}$
$=k\sin c\sin\text{A}\cos\text{B}$,

$\therefore$ $(\cos a+\cos b)(1-\cos c)=k\sin c\sin(\text{A}+\text{B})$ [add
$=(\sin a\pm\sin b):(\sin\text{A}\pm\sin\text{B})\cdot\sin c\sin(\text{A}+\text{B})$,

$\therefore$ $(\sin a\pm\sin b):(\cos a+\cos b)$
$=(\sin\text{A}\pm\sin\text{B}):\sin(\text{A}+\text{B})\cdot(1-\cos c):\sin c$;

i.e., $2\sin\tfrac{1}{2}(a\pm b)\cos\tfrac{1}{2}(a\mp b):2\cos\tfrac{1}{2}(a+b)\cos\tfrac{1}{2}(a-b)$
$=2\sin\tfrac{1}{2}(\text{A}\pm\text{B})\cos\tfrac{1}{2}(\text{A}\mp\text{B}):2\sin\tfrac{1}{2}(\text{A}+\text{B})\cos\tfrac{1}{2}(\text{A}+\text{B})\cdot\tan\tfrac{1}{2}c$,
[II th. 12

$\therefore$ $\tan\tfrac{1}{2}(a+b)=\cos\tfrac{1}{2}(\text{A}-\text{B}):\cos\tfrac{1}{2}(\text{A}+\text{B})\cdot\tan\tfrac{1}{2}c$

and $\tan\tfrac{1}{2}(a-b)=\sin\tfrac{1}{2}(\text{A}-\text{B}):\sin\tfrac{1}{2}(\text{A}+\text{B})\cdot\tan\tfrac{1}{2}c$. Q.E.D.

Prove the formulæ for $\tan\tfrac{1}{2}(\text{A}+\text{B})$, $\tan\tfrac{1}{2}(\text{A}-\text{B})$ as the polar formulæ of those for $\tan\tfrac{1}{2}(a+b)$, $\tan\tfrac{1}{2}(a-b)$.

COR. 1. *In any spherical triangle, whose parts are all positive and less than two right angles, the half-sum of any two sides and the half-sum of their opposite angles are of the same species.*

For $\because$ $\tan\tfrac{1}{2}(\text{A}+\text{B})\cos\tfrac{1}{2}(a+b)=\cos\tfrac{1}{2}(a-b)\cot\tfrac{1}{2}\text{C}$,
a positive number,

$\therefore$ $\tan\tfrac{1}{2}(\text{A}+\text{B})$, $\cos\tfrac{1}{2}(a+b)$ are both positive or both negative, or the first is infinite and the second zero,

$\therefore$ $\tfrac{1}{2}(\text{A}+\text{B})$, $\tfrac{1}{2}(a+b)$ are both acute, or both obtuse, or both right. Q.E.D.

DELAMBRE'S FORMULÆ.

COR. 2. $\sin\frac{1}{2}(A+B) = \pm\cos\frac{1}{2}(a-b) : \cos\frac{1}{2}c\cdot\cos\frac{1}{2}C$,

$\cos\frac{1}{2}(A+B) = \pm\cos\frac{1}{2}(a+b) : \cos\frac{1}{2}c\cdot\sin\frac{1}{2}C$,

$\sin\frac{1}{2}(A-B) = \pm\sin\frac{1}{2}(a-b) : \sin\frac{1}{2}c\cdot\cos\frac{1}{2}C$,

$\cos\frac{1}{2}(A-B) = \pm\sin\frac{1}{2}(a+b) : \sin\frac{1}{2}c\cdot\sin\frac{1}{2}C$.

For, $\because$ $(\cos a + \cos b)(1-\cos c) = k \sin c \sin(A+B)$ [above

$= \sin c : \sin C\cdot\sin c \sin(A+B)$,

$\therefore$ $\sin(A+B) = (\cos a + \cos b)(1-\cos c)\sin C : \sin^2 c$,

$= 2\cos\frac{1}{2}(a+b)\cos\frac{1}{2}(a-b)\, 2\sin^2\frac{1}{2}c \sin C : 4\sin^2\frac{1}{2}c\cos^2\frac{1}{2}c$ [II, ths. 12, 13

$= \cos\frac{1}{2}(a+b)\cos\frac{1}{2}(a-b)\sin C : \cos^2\frac{1}{2}c$;

and $\because$ $\tan\frac{1}{2}(A+B) = \cos\frac{1}{2}(a-b) : \cos\frac{1}{2}(a+b)\cdot\cot\frac{1}{2}C$ [Nap. anal.

and $\sin(A+B)\tan\frac{1}{2}(A+B) = 2\sin^2\frac{1}{2}(A+B)$,

and $\sin C\cot\frac{1}{2}C = 2\cos^2\frac{1}{2}C$,

$\therefore$ $\sin^2\frac{1}{2}(A+B) = \cos^2\frac{1}{2}(a-b) : \cos^2\frac{1}{2}c\cdot\cos^2\frac{1}{2}C$. Q. E. D.

$\therefore$ $\sin\frac{1}{2}(A+B) = \pm\cos\frac{1}{2}(a-b) : \cos\frac{1}{2}c\cdot\cos\frac{1}{2}C$.

The $\left\{\begin{matrix}\text{second}\\\text{third}\end{matrix}\right.$ of Delambre's formulæ comes readily from the first (just proved) by dividing the first, member by member, by Napier's $\left\{\begin{matrix}\text{third}\\\text{second}\end{matrix}\right.$ analogy [th. 9]; and Delambre's fourth formula, by dividing the second by Napier's first analogy, or the third by his fourth analogy, or as the polar of the first formula; or the three may be got in the same way as the first.

Since Delambre's formulæ may be got by division, as above, their second members must all be taken positive or all negative.

If all the parts of a triangle be positive and less than two right angles, all the radicals must be taken positive, since each member of the first formula is positive.

In general, if the radicals be positive for a given triangle, they are negative for a triangle whose parts are the same as those of the other, except that one side or one angle differs from the corresponding side or angle by four right angles.

EXAMPLES.

1. From the two equations

$$\cos b = \cos c \cos a + \sin c \sin a \cos \text{B},$$
$$\cos c = \cos a \cos b + \sin a \sin b \cos \text{C},$$

eliminate $\cos c$, and prove that

$$\sin a \cos b = \cos a \sin b \cos \text{C} + \sin c \cos \text{B};$$

thence eliminate $\sin c$, and prove that

$$\sin a \cot b = \cos a \cos \text{C} + \cot \text{B} \sin \text{C};$$

and write the five symmetric formulæ.

2. Make C a right angle and from the formulæ of ths. 6, 7, and from those of ex. 1, find the formulæ of th. 5.

Show which of the corollaries of th. 5 are particular cases of the corollaries of ths. 6, 7.

3...15. In a spherical triangle ABC, whose parts are all positive and less than two right angles, prove that:

3. $\sin s : \sin c = \cos \frac{1}{2}\text{A} \cos \frac{1}{2}\text{B} : \sin \frac{1}{2}\text{C}$.

4. $\cos \text{S} : \sin \text{C} = - \sin \frac{1}{2}a \sin \frac{1}{2}b : \cos \frac{1}{2}c$.

5. $\sin (s - c) : \sin c = \sin \frac{1}{2}\text{A} \sin \frac{1}{2}\text{B} : \sin \frac{1}{2}\text{C}$.

6. $\cos (\text{S} - \text{C}) : \sin \text{C} = \cos \frac{1}{2}a \cos \frac{1}{2}b : \cos \frac{1}{2}c$.

7. $\sin (s - a) : \sin c = \cos \frac{1}{2}\text{A} \sin \frac{1}{2}\text{B} : \cos \frac{1}{2}\text{C}$.

8. $\cos (\text{S} - \text{A}) : \sin \text{C} = \sin \frac{1}{2}a \cos \frac{1}{2}b : \sin \frac{1}{2}c$.

9. $\sin (s - c) : \sin (s - a) = \tan \frac{1}{2}\text{A} : \tan \frac{1}{2}\text{C}$.

10. $\cos (\text{S} - \text{C}) : \cos (\text{S} - \text{A}) = \cot \frac{1}{2}a : \cot \frac{1}{2}c$.

11. $\sin (s - a) : \sin s = \tan \frac{1}{2}\text{B} \tan \frac{1}{2}\text{C}$.

12. $\cos (\text{S} - \text{A}) : \cos \text{S} = - \cot \frac{1}{2}b \cot \frac{1}{2}c$.

13. $\sin (s - a) \tan \frac{1}{2}\text{A} = \sin (s - b) \tan \frac{1}{2}\text{B} = \sin (s - c) \tan \frac{1}{2}\text{C}$.

14. $\cos (\text{S} - \text{A}) \cot \frac{1}{2}a = \cos (\text{S} - \text{B}) \cot \frac{1}{2}b = \cos (\text{S} - \text{C}) \cot \frac{1}{2}c$.

15. $\text{S} - \text{A},\ \text{S} - \text{B},\ \text{S} - \text{C} < 90°$.

§ 6. SOLUTION OF OBLIQUE SPHERICAL TRIANGLES.

PROB. 2. TO SOLVE AN OBLIQUE SPHERICAL TRIANGLE.

Apply such of the formulæ of ths. 6–9 as serve to express the values of the three unknown parts in terms of three known parts.

CHECK: *form an equation involving the three computed parts and such of the given parts as may be necessary.*

Delambre's formulæ are useful as checks; and so are the formulæ of exs. 3–14, § 5. In no case may the same parts be used in the same way in the solution and the check.

THE PARTS ALL POSITIVE AND LESS THAN TWO RIGHT ANGLES.

(a) *Given* b, c, A, *two sides and the included angle:*

Find $\frac{1}{2}(\mathrm{B}+\mathrm{C})$, $\frac{1}{2}(\mathrm{B}-\mathrm{C})$, [Nap. anal.

then $\frac{1}{2}(\mathrm{B}+\mathrm{C})+\frac{1}{2}(\mathrm{B}-\mathrm{C})=\mathrm{B}$, $\frac{1}{2}(\mathrm{B}+\mathrm{C})-\frac{1}{2}(\mathrm{B}-\mathrm{C})=\mathrm{C}$.

and $\sin a=\sin b:\sin \mathrm{B}\cdot\sin \mathrm{A}$. [law of sines

There is always one triangle, and but one; the parts B, C are found without ambiguity by the formulæ, and by th. 9 cr. 1, and the species of a is determined by Napier's analogies.

(b) *Given* B, C, a, *two angles and the included side:*

Find $\frac{1}{2}(b+c)$, $\frac{1}{2}(b-c)$, [Nap. anal.

then $\frac{1}{2}(b+c)+\frac{1}{2}(b-c)=b$, $\frac{1}{2}(b+c)-\frac{1}{2}(b-c)=c$,

and $\sin \mathrm{A}=\sin \mathrm{B}:\sin b\cdot\sin a$.

There is always one triangle, and but one; the parts b, c are found without ambiguity by the formulæ, and by th. 9 cr. 1, and the species of A is determined by Napier's analogies.

The triangle may also be solved by applying the rules for (a) to the polar triangle.

(c) *Given* a, b, A, *two sides and an angle opposite one of them:*

then $\sin \mathrm{B}=\sin \mathrm{A}:\sin a\cdot\sin b$.

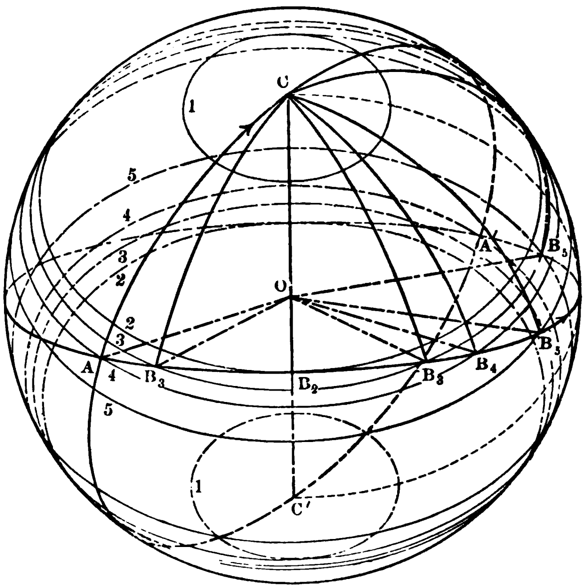
Find c, C. [Nap. anal.

If a, b, A be all right, the triangle is biquadrantal, and C, c are indeterminate and equal.

If a, b, A be not all right, there are five cases:

Let ABA′ be any great circle of a sphere, AA′ a diameter of the sphere, CC′ any diameter not meeting ABA′;

draw the great circle CAC′A′;



then the angle BAC is a fixed angle, and AC is a fixed arc.

With C as pole and with different arc-radii draw small circles as follows:

the circles 1, 1 not to cut AB;

the circles 2, 2 to touch AB;

the circles 3, 3 to cut AB in two points, to the right of A and to the left of A′;

the circles 4, 4 to cut AB at A and a point to the right of A, and at A′ and a point to the left of A′;

the circles 5, 5 to cut AB to the left and right of A, and to the right and left of A′.

Let p be the arc-ordinate of C as to AB.

In the figure as drawn and described, A, b are both acute; but with the arcs properly directed all possible cases may be illustrated by the same figure; *i.e.*, A may be acute and b obtuse, or A obtuse and b acute, or A, b both obtuse.

1. *a less near right than p:*

then $\because$ $\sin B = \sin p : \sin a > 1$,

$\therefore$ there is no triangle.

The arc a, swinging from the point C, does not reach, or it overreaches, the base.

2. *a just as near right as p:*

then $\because$ $\sin B = \sin p : \sin a = 1$,

$\therefore$ B is a right angle;

and $\because$ b is nearer right than p and than a, [th. 5 cr. 2

$\therefore$ if A, a be of $\left\{ \begin{matrix} \text{the same} \\ \text{opposite} \end{matrix} \right.$ species, there is $\left\{ \begin{matrix} \text{one} \\ \text{no} \end{matrix} \right.$ triangle.

The arc a reaches the base at one point and subtends the angle A or the supplement of A.

3. *a nearer right than p, but less near right than b:*

then $\because$ $\sin B < 1$, and b is nearer right than a,

$\therefore$ if A, a be of $\left\{ \begin{matrix} \text{the same} \\ \text{opposite} \end{matrix} \right.$ species, there are $\left\{ \begin{matrix} \text{two} \\ \text{no} \end{matrix} \right.$ triangles. [th. 6 cr. 1

The arc a cuts the base at B_3, B'_3 on the same side of A.

4. *a just as near right as b:*

then $\because$ $\sin B < 1$, and B, b are of the same species, [th. 6 cr. 1

$\therefore$ if A, a be of $\left\{ \begin{matrix} \text{the same} \\ \text{opposite} \end{matrix} \right.$ species, there is $\left\{ \begin{matrix} \text{one} \\ \text{no} \end{matrix} \right.$ triangle.

The arc a cuts the base at B_4 and at A or A$'$.

5. *a nearer right than b :*

then $\because$ $\sin B < 1$, and B, b are of the same species, [th. 6 cr. 1

$\therefore$ there is but one triangle.

The arc a cuts the base at two points, on opposite sides of A.

(*d*) *Given* A, B, *a*, *two angles and a side opposite one of them:* then $\sin b = \sin a : \sin A \cdot \sin B$.

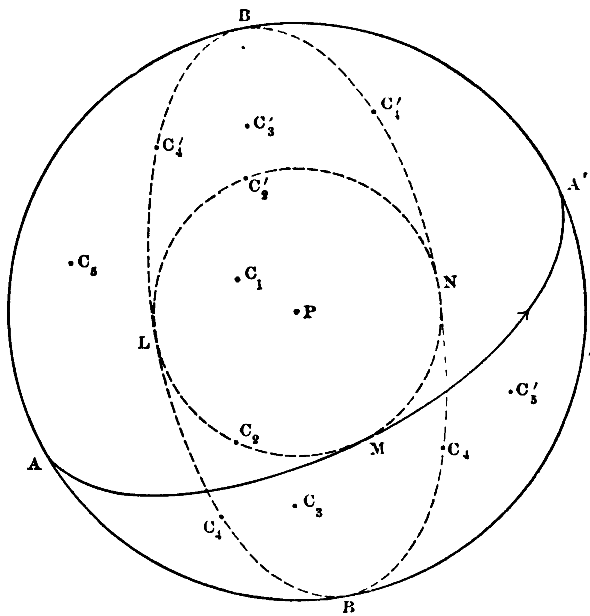
Find *c*, C. . [Nap. anal.

If A, B, *a* be all right, the triangle is biquadrantal and *c*, C are indeterminate and equal. [th. 5 cr. 1

If A, B, *a* be not all right, there are five cases.

Let ABA′B′ be any great circle of a sphere, whose pole is P, and with P as pole and arc-radius PL, equal to the complement of A, draw a small circle LMN;

let angle A slide along the great circle AB;



then the terminal arc of A is always tangent to the circle LMN, as at BLB′, AMA′, B′NB.

As the figure is drawn A is acute, B is the vertex of the angle B, and the terminal arc of B falls within, upon, or without the

arc BB′N according as A is less near, just as near, or nearer right than B.

If C lie within B′LMN, or upon the arcs LB′, B′N, there is no triangle;

if C lie without the arc BB′N, or upon the arcs BL, LMN, NB, there is one triangle;

if C lie within BLMN, there are two triangles.

So, with the arcs properly directed, if A be obtuse, and B′ be the vertex of the angle B.

Let p be arc-ordinate of C as to AB, so that $\sin p = \sin b \sin A$;

1. A *less near right than p:*

then $\because$ $\sin b = \sin p : \sin A > 1$, [th. 4, hyp.

$\therefore$ there is no triangle.

The point A slides along AB as base, and the terminal arc of A does not reach, or it overreaches, the vertex C.

2. A *just as near right as p:*

then $\because$ $\sin b = \sin p : \sin A = 1$,

$\therefore$ b is a right arc;

and $\because$ B is nearer right than p and than A [th. 5 cr. 2

$\therefore$ if a, A be of $\left\{\begin{matrix}\text{the same}\\ \text{opposite}\end{matrix}\right.$ species, there is $\left\{\begin{matrix}\text{one}\\ \text{no}\end{matrix}\right.$ triangle. [th. 7 cr. 1

The terminal arc of A passes through the vertex C once, and A is subtended by arc a or the supplement of a.

3. A *nearer right than p, but less near right than* B:

then $\because$ $\sin B < 1$, and B is nearer right than A, [hyp.

$\therefore$ if a, A be of $\left\{\begin{matrix}\text{the same}\\ \text{opposite}\end{matrix}\right.$ species, there are $\left\{\begin{matrix}\text{two}\\ \text{no}\end{matrix}\right.$ triangles.

The terminal arc of A passes through the vertex C twice, and A is subtended twice by arc a, or twice by the supplement of a.

4. A *just as near right as* B:

then $\because$ $\sin B < 1$, and b, B are of the same species, [th. 7 cr. 1

$\therefore$ if a, A be of $\left\{\begin{matrix}\text{the same}\\ \text{opposite}\end{matrix}\right.$ species, there is $\left\{\begin{matrix}\text{one}\\ \text{no}\end{matrix}\right.$ triangle.

The terminal arc of A passes through the vertex C twice: once when A coincides with B or B′, the point opposite B on the sphere, and once when A is subtended by arc a or the supplement of a.

5. A *nearer right than* B:

then $\because$ $\sin B < 1$, and B, b are of the same species, [th. 7 cr. 1

$\therefore$ there is one triangle.

The terminal arc of A passes through the vertex C twice: in the one position A is subtended by arc a, in the other by the supplement of a.

The triangle may also be solved by applying the rules for (c) to the polar triangle.

(e) *Given a, b, c, the three sides:*

Apply the formulæ of th. 6.

If the half sum of the sides be not greater than either side and less than two right angles, there is no triangle; otherwise there is one triangle, and but one.

That there is a single triangle appears also from the formulæ: for the half-angles computed must be less than a right angle, and but one such half-angle can be found from a given function.

Of these formulæ those of cr. 2 use nine different logarithms, those of cr. 3 use ten different logarithms, and those of cr. 4 use only seven different logarithms, for the computation of all the angles. Those of cr. 4 are, therefore, generally to be preferred; they may be put in the form:

$$\tan \tfrac{1}{2} A = \frac{1}{\sin(s-a)} \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}},$$

$$\tan \tfrac{1}{2} B = \frac{1}{\sin(s-b)} \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}},$$

$$\tan \tfrac{1}{2} C = \frac{1}{\sin(s-c)} \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}};$$

wherein the second factor of the right member is the same, and may be computed once, for all.

(*f*) *Given* A, B, C, *the three angles:*

Apply the formulæ of th. 7.

If the half spherical excess be not greater than zero and less than either angle, there is no triangle; otherwise there is one triangle, and but one.

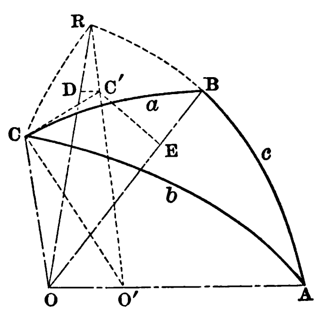
That there is a single triangle appears also from the formulæ: for the half-sides computed must be less than a quadrant, and but one such half-side can be found from a given function.

Among the formulæ of th. 7 the same choice may be made as in (*e*); the reader may transform those of cr. 4.

The triangle may also be solved by applying the rules for (*e*) to the polar triangle.

GRAPHIC SOLUTION OF SPHERICAL TRIANGLES.

Let ABC be a spherical triangle and O the centre of the sphere; with A as pole draw a small circle through C cutting arc AB in R; project R on OA at O′ and C on O′R at C′;



through C′ draw a parallel to OA cutting OR in D;

then $\because$ CC′ $\perp$ plane OAR and O′C, O′C′ $\perp$ OA, [geom.

$\therefore$ OD : OR = O′C′ : O′R = O′C′ : O′C = cos A,

$\therefore$ OD = OR cos A.

Project C′ on OB at E;

then $\because$ CE $\perp$ OB, [geom.

$\therefore$ OE = OC cos a = OR cos a.

(*a*) *Given* a, b, c:

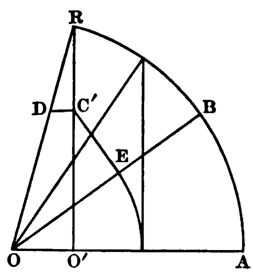
from any centre O, and in the same plane, draw equal rays OA, OB, OR, making ∠s AOR, AOB $= b, c$;

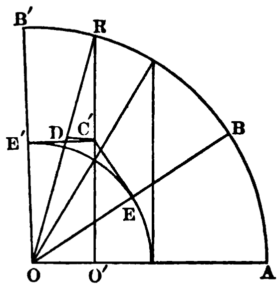
on OB take E such that OE = OR cos a;

project R on OA at O′ and at E draw a perpendicular to OB meeting O′R at C′;

through C′ draw a parallel to OA cutting OR in D;

then cos A = OD : OR; and so for cos B, cos C.





(*b*) *Given* A, b, c:

construct ∠s AOR, AOB as above;

on OR take D such that OD = OR cos A;

project R on OA at O′;

through D draw a parallel to OA cutting O′R in C′;

project C′ on OB at E;

then cos a = OE : OR.

Find cos B, cos C, as in (*a*).

(*c*) *Given* A, a, b:

construct ∠ AOR as above;

on OR take D such that OD = OR cos A;

project R on OA at O′;

through D draw a parallel to OA cutting O′R in C′;

with centre O and radius OR cos a draw a circle;

and from C′ draw tangents to this circle at E, E′;

take ∠s AOE, AOE′ as c, c';

then, if $oc' \begin{cases} > \\ = \\ < \end{cases}$ OR $\cos a$, there are $\begin{cases} \text{two} \\ \text{one} \\ \text{no} \end{cases}$ triangles.

(*d*) (*e*) (*f*) *Given* $\begin{cases} A, B, C; \\ a, B, C; \\ a, A, B; \end{cases}$ solve the polar triangle $\begin{cases} a'b'c' \\ A'b'c' \\ A'a'b' \end{cases}$ as in (*a*) (*b*) (*c*)

EXAMPLES.

1...10. Solve the oblique triangles, and check the work, given:

1. $b = 98° 12'$, $c = 80° 35'$, $A = 10° 16'$.
   [149° 32′ 51″, 30° 20′ 29″, 20° 22′ 7″

2. $A = 135° 15'$, $C = 50° 30'$, $b = 69° 34'$.
   [50° 6′ 16″, 120° 41′ 47″, 70° 28′ 9″

3. $a = 40° 16'$, $b = 47° 14'$, $A = 52° 30'$.

4. $a = 120°$, $b = 70°$, $A = 130°$.
   [58° 57′ 20″, 75° 36′ 4″, 56° 13′ 23″
   or 165° 23′ 44″, 163° 26′ 16″, 123° 46′ 37″

5. $a = 40°$, $b = 50°$, $A = 50°$.
   [65° 54′ 52″, 82° 48′ 42″, 56° 21′ 24″
   or 114° 5′ 8″, 22° 16′ 52″, 18° 33′ 2″

6. $A = 132° 16'$, $B = 139° 44'$, $a = 127° 30'$.
   [136° 8′ 16″, 114° 17′ 48″, 77° 43′ 4″

7. $A = 110°$, $B = 60°$, $a = 50°$.

8. $A = 70°$, $B = 120°$, $a = 80°$.
   [114° 49′ 26″, 65° 48′ 58″, 72° 56′ 48″

9. $a = 100°$, $b = 50°$, $c = 60°$.
   [138° 15′ 45″, 31° 11′ 14″, 35° 49′ 58″

10. $A = 120°$, $B = 130°$, $C = 80°$.
    [144° 10′ 2″, 148° 48′ 46″, 41° 44′ 15″

11. Test the results found above by the graphic solution of the triangles.

*§ 7. SPHERICAL ASTRONOMY.

THE CELESTIAL SPHERE.

In astronomy the elements of position of a heavenly body are distance and direction; in spherical astronomy only one of these elements, direction, is regarded, and that is usually referred to the earth's centre. For this purpose all stars may be considered as at the same distance from the earth's centre upon the surface of a sphere of arbitrary radius called the *celestial sphere*.

The trace of the plane of the earth's equator on this sphere is the *celestial equator*, whose poles (north and south) are the traces of the earth's axis.

The *ecliptic* is a great circle of the celestial sphere, the sun's apparent path in one year due to the motion of the earth around the sun; it cuts the equator in two points, the *vernal* and the *autumnal equinox*, which are passed through by the sun, on March 20 and September 23. The *obliquity* of the ecliptic is the nearly constant angle of 23° 27′ between the planes of the ecliptic and equator.

Secondaries to any great circle, or *primary*, are great circles cutting it, and therefore its parallels at right angles. Secondaries to the celestial equator are *hour-circles*.

To any observer the *sensible horizon* is a plane touching the earth's surface at the point of observation; and a plane parallel to this plane through the earth's centre traces out on the celestial sphere the *rational horizon*, whose poles, *zenith* and *nadir*, are the traces of a vertical line, and whose secondaries are *vertical circles*. One of the vertical circles is also an hour-circle, the observer's *celestial meridian*, and passes through his zenith and nadir, and the north and south poles of the celestial sphere; its plane is the same with that of the observer's terrestrial meridian, and it meets the plane of his sensible horizon in his *meridian line*. The vertical circle that is perpendicular to the meridian is the observer's *prime vertical*, and it goes through the east and west points of his horizon.

SPHERICAL COORDINATES.

As the position of a point on the earth's surface is defined by means of two coordinates (latitude and longitude), the standards of reference being a convenient great circle (the equator) and a convenient point on it (the point where it is crossed by the meridian of Greenwich); so, the position of a star at any instant on the celestial sphere may be defined in either of three ways:

1. *As to the celestial equator:*

The *declination* of a star is its angular distance (north or south) from the celestial equator measured upon its hour-circle; and the arc of the equator intercepted between this circle and the vernal equinox is the star's *right ascension;* it is reckoned eastward from the vernal equinox from 0° to 360°. The complement of the declination is the polar distance.

Instead of a star's right ascension its *hour-angle* is often used, in problems that involve diurnal motion, to define its hour-circle at any instant; this angle is the angle at the pole between the observer's celestial meridian and the star's hour-circle, and is counted from the meridian, positive towards the west and negative towards the east. The right ascension of a fixed star changes very little, since the vernal equinox is nearly fixed on the celestial sphere; the hour-angle changes every moment.

2. *As to the ecliptic:*

The *latitude* of a star is its angular distance from the ecliptic measured on a secondary; and the arc of the ecliptic intercepted between the vernal equinox and this secondary, measured eastward, is the star's *longitude.*

3. *As to the horizon:*

The *altitude* of a star is its angular distance from the horizon measured on a vertical circle; and the arc of the horizon intercepted between this circle and the south point of the horizon is the star's *azimuth.* Owing to the rotation of the celestial sphere, the horizon-coordinates change every moment.

RELATIONS BETWEEN ECLIPTIC-COORDINATES AND EQUATOR-COORDINATES.

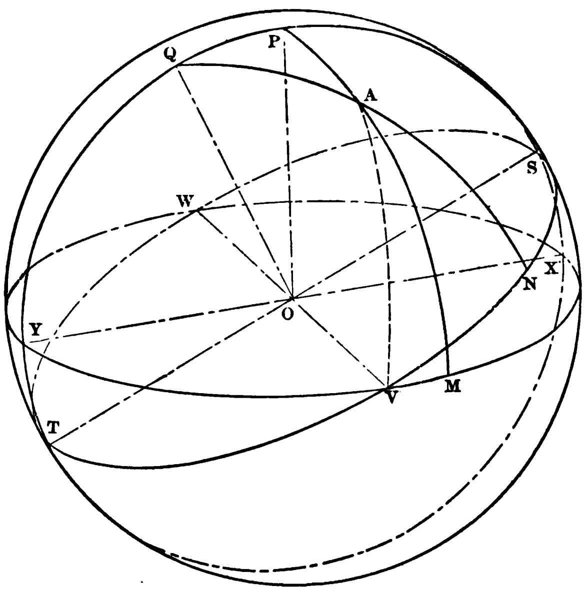
On the celestial sphere let P be the pole of the equator, Q the pole of the ecliptic;

then the great circle through P, Q is the common secondary of the equator and the ecliptic;

let V, W be the vernal and the autumnal equinox at quadrantal distances from S, T;

let PAM be the hour-circle of a star A and QAN the secondary to the ecliptic;

then VM, MA are the right ascension and declination of A, and VN, NA are its longitude and latitude.



These four coordinates of any fixed star are subject to only slight variations in any one year; they are recorded for the principal stars in a yearly almanac, with the data for com-

puting the variations; the sun's declination is recorded for each day or half day, and may be got for any hour and minute by interpolation.

The spherical angle XVS is the obliquity of the ecliptic, and, since VX, VS are quadrants, XVS is measured by the arc XS;

arcs XS, PQ, YT are each 23° 27′, and arcs SP, YQ are each 66° 33′.

Equator-coordinates may be converted into ecliptic-coordinates:

when VM, MA are given in the right spherical triangle MVA, the arc VA and the angle MVA may be found;

the angle NVA is found by subtracting the obliquity, and the triangle NVA may be solved for VN, NA; so conversely.

THE SUN'S ANNUAL MOTION.

The particular case of the sun is simpler: since his apparent annual path is the ecliptic, his latitude is always zero, and his right ascension, declination, and longitude are the arc-abscissa, arc-ordinate, and arc-distance of a given angle, the obliquity; his declination increases from 0° at V on March 20 to 23° 27′ at S on June 21 (the summer solstice), then decreases to 0° at W on September 23, and to ⁻23° 27′ at T on December 22 (the winter solstice), then increases to 0° at V; his right ascension and longitude are equal at 0°, 90°, 180°, 270°, 360°.

EXAMPLES.

1. The altitude of a circumpolar star at upper transit is 60° and at lower transit 40°; find the declination of the star.

2. The vernal equinox culminated at $0^h\ 10^m\ 13^s$, and a certain star culminated at $2^h\ 5^m\ 10^s$; find its right ascension.

3. Find the latitude and longitude of a star whose right ascension is $5^h\ 13^m$ and declination 60°.

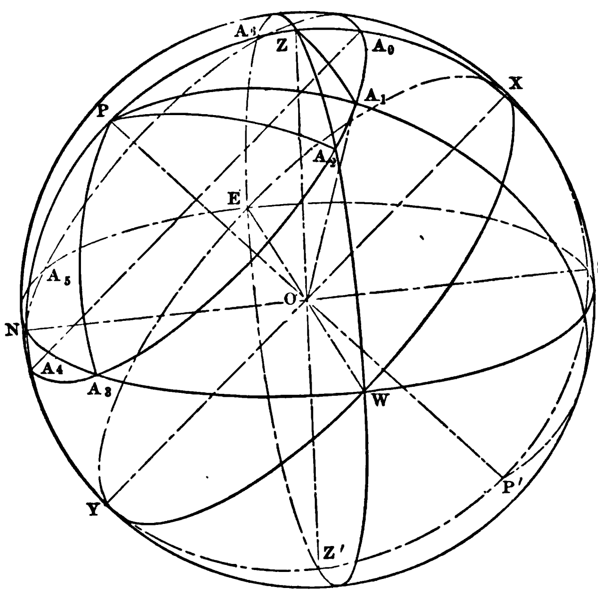
4. When the sun's declination is 15°, find his right ascension and longitude.

RELATIONS BETWEEN EQUATOR-COORDINATES AND HORIZON-COORDINATES. — THE ASTRONOMICAL TRIANGLE.

On the celestial sphere let P be the pole of the equator XY, and Z that of the horizon NS;

then the great circle through PZ is the celestial meridian, the common secondary of equator and horizon;

let ZWZ′E be the prime vertical perpendicular to both meridian and horizon and meeting both equator and horizon in the east and west points.



The celestial sphere appears to make a complete revolution on its axis PP′ in about $23^h\,56^m\,4^s$ of civil time. This is the interval between two successive transits of any fixed star, and is a *sidereal day*. A sidereal clock shows 0 hours when the vernal equinox culminates; and the hours are marked from 0 to 24. The sidereal time of a star's transit gives its exact right

ascension, which may be converted into angular measure at the rate of 15° to a sidereal hour, or 1° to 4 minutes, 15′ to 1 minute, 1′ to 4 seconds, and so on.

The hour-circle of the star A coincides with the meridian in the position PA_0 bearing due south as seen from O, and the star has then its greatest altitude:

in the position PA_2 the star is on the prime vertical and bears due west;

in the position PA_3 the star sets below the horizon;

it reaches its greatest depression at A_4 when its hour-circle passes over the meridian bearing due north;

it rises at A_5, reaches the prime vertical at A_6 bearing due east, and culminates again at A_0.

The spherical triangle ZPA_1 for any position of the star A is the *astronomical triangle:*

its sides ZA_1, PA_1 are the co-altitude and co-declination of A_1;

the angles ZPA_1, PZA_1 are the supplement of the hour-angle and of the azimuth of A;

and the side PZ is the observer's co-latitude;

for $\because$ this co-latitude is the angle between the earth's axis and the vertical line at the point of observation,

and the traces of these lines on the celestial sphere are P, Z,

$\therefore$ the arc PZ measures the observer's co-latitude.

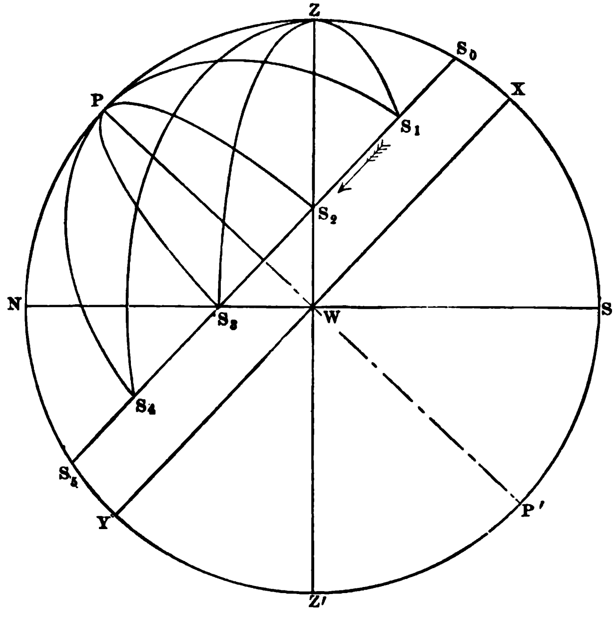
When the latitude is known the relations between the sides and angles of this triangle give the relations between the star's equator- and horizon- coordinates.

The observer's latitude may be determined, once for all, by the astronomical triangle when the declination, the altitude, and either the azimuth or the hour-angle of a heavenly body are known for some instant. If at the time of observation the body be on the meridian the hour-angle is zero, and the azimuth either zero or 180°; if it be on the prime vertical, the azimuth is $\pm 90°$; if it be on the horizon, the altitude is zero; and in all these cases the computation of latitude is simplified.

THE SUN'S DIURNAL MOTION. — SOLAR TIME.

The sun's hour-circle PS_0 coincides with the observer's meridian at noon; the hour-angle ZPS_1 at any instant measures the time of observation from noon; the angle ZPS_3 measures the time of sunset.

The greater the declination XS_0, the greater is the hour-angle of setting, ZPS_3, the longer the day, and the shorter the night. The day is longest in the northern hemisphere when the declination is greatest, June 21, the summer solstice. When the



declination is zero, the diurnal path S_0S_5 coincides with the equator, and is bisected by the horizon; the day is then equal in duration to the night, and hence the term equinox. When the declination is 23° 27′ s., the day is shortest in north latitudes, and the night longest (winter solstice).

The interval between two successive transits of the sun over the same meridian is an *apparent solar day*. This interval varies, from two causes: the obliquity of the ecliptic, and the variability of the sun's apparent motion in the ecliptic.

The *mean sun* is an imaginary body, supposed to move uniformly in the equator with the annual period, and with the average velocity, of the true sun. It culminates at *civil* or *mean noon*, and the constant interval between two successive transits is a *mean solar day*. This interval is divided into hours, minutes, and seconds. A second of mean solar time is the ordinary time-unit, and is the same fraction of a mean solar day as a sidereal second is of a sidereal day. The $\left\{\begin{matrix}\textit{mean}\\ \textit{apparent}\end{matrix}\right.$ *solar time* is the hour-angle of the $\left\{\begin{matrix}\text{mean}\\ \text{true}\end{matrix}\right.$ sun, at any instant. The angle between the mean and true hour-circles is recorded for each day, in the almanac, as the *equation of time*. It varies throughout the year between 0 and about ± 16 minutes of time.

The *astronomical day* begins at mean noon, and the hours are numbered from 0 to 24.

In what follows, apparent time is used.

EXAMPLES.

5. The meridian altitude of the sun's centre was $25° 38' 30''$ s., and his declination $22° 18' 14''$ s.; find the latitude.

6. The meridian altitude of Jupiter was $50° 20' 8''$ s., and his declination $18° 47' 37''$ N.; find the observer's latitude.

7. The sun crossed the prime vertical at an altitude of $54°$; find the observer's latitude and the time of day, the sun's declination, got by interpolation for the approximate time of day, being $18° 30'$.

   Here, $ZS_2 = 36°$, $PS_2 = 71° 30'$, $PZS_2 = 90°$; find ZP, ZPS_2.

8. Find the observer's latitude in ex. 1.

   In what latitude will this star just graze the horizon?

9. If the sun's declination be 22° 26′ N., altitude 40° 55′ at 3 P.M., find the observer's latitude.

   In this example, $ZS_1 = 49° 5'$, $PS_1 = 67° 34'$, $ZPS_1 = 3$ h. $= 45°$, and the co-latitude, PZ, is to be found.

10. In latitude 13° 17′ N. the sun's altitude was 36° 37′, his declination was 22° 10′ S.; find his hour-angle.

11. If the sun's declination be 17° N., find the time in the afternoon when he will be due west from a place in latitude 51° N; and find how far from the west point he will set (his *amplitude* at setting).

12. If the sun be due west at setting (amplitude zero), find his declination and the time of year.

13. If the time of sunset be sought on any given day, a quadrantal triangle PZS_3 may be solved for the hour-angle ZPS_3.

    If the sun's declination be 14° S. and the latitude 42° N., find the time and amplitude of sunrise and sunset.

14. Find the time of setting in ex. 12.

15. Find the length of the longest day in Ithaca, excluding twilight, latitude 42° 30′ N.

16. Find the lowest north latitude in which the sun does not set on the longest day, nor rise on the shortest day.

17. Find the time of sunrise in Boston, latitude 42° 21′ N., on the shortest day of the year, and the sun's amplitude.

18. The phenomenon of twilight is due to the reflection and refraction of some of the sun's rays toward the observer's eye when the direct rays are intercepted; it begins or ends when the sun is about 18° below the horizon:

    How long does twilight last in Boston on the shortest day?

    Given $ZS_4 = 90° + 18°$, PS_4, ZP; find ZPS_4, and subtract the hour-angle of sunset, ZPS_3.

19. Find the length of the longest day in Ithaca, including morning and evening twilight.

20. In what latitude does the sun get just 18° below the horizon on the longest day, so that twilight lasts all night?

Here, $NS_5 = 18°$, $PS_5 = 66° 33'$; find the co-latitude ZP.

21. Given the declination of Aldebaran, 16° 17′ N; find his altitude and azimuth to an observer at Boston when the hour-angle of this star is $3^h 25^m 12^s$; and the hour-angle and amplitude at rising and setting.

22. Find the angular distance between the moon, altitude 40°, azimuth 25° W., and Venus, altitude 24°, azimuth 90° W.

23. At what time of day, March 20, do shadows in Ithaca point E.N.E.?

24. The greatest altitude of a star was 40° in latitude 50° N.; find its declination.

*§ 8. NAVIGATION.

When a mariner cannot make celestial observations, he has recourse to *dead-reckoning*, *i.e.*, he computes the position of his ship from the latitude and longitude of her starting-point, or of the place of last observation, and the records of sailing. This dead-reckoning is the subject of navigation proper, as distinguished from nautical astronomy.

The rate of sailing is usually recorded every hour, and is measured by the *log-line*. This is a line wound on a reel and attached to a small quadrantal piece of board. The quadrant is loaded on the arc with lead to keep it upright when thrown into the water and prevent its moving forward toward the ship while the line is running out. The log-line is divided into *knots*, each a hundred-twentieth part of a nautical mile, so that the number of knots run out in half a minute gives the ship's hourly rate in miles.

The direction of sailing at any time is shown by the *mariner's compass*.

The reading of the compass is to be corrected for variation, deviation, and leeway.

The *variation* is the angle between the magnetic and true meridians; it is found, for various places, by astronomical observations, and laid down on the nautical charts.

The *deviation* is the angle of deflection of the needle from the magnetic meridian, caused by the iron of the ship; it is found, for a given ship and a given direction, by special experiments.

When there is a side wind, the angle which the ship's track makes with her fore-and-aft line is the *leeway;* it is found, for a given ship, a given freight, and a given obliquity and velocity of the wind, by special experiments.

The corrected reading is the *course;* it is the angle between the ship's true meridian and her true direction of motion. In what follows, the corrections are supposed to have been made, so that the given courses are the true courses. When the course is kept constant, the ship's track crosses every meridian at the same angle; the path is neither straight nor circular, but a spiral, the *loxodrome* or *rhumb-line*, that goes round and round the earth's surface, coming nearer and nearer to the pole; and its length is the *distance.*

The meridian length between the first and last parallel of latitude is the *difference of latitude* made by the ship.

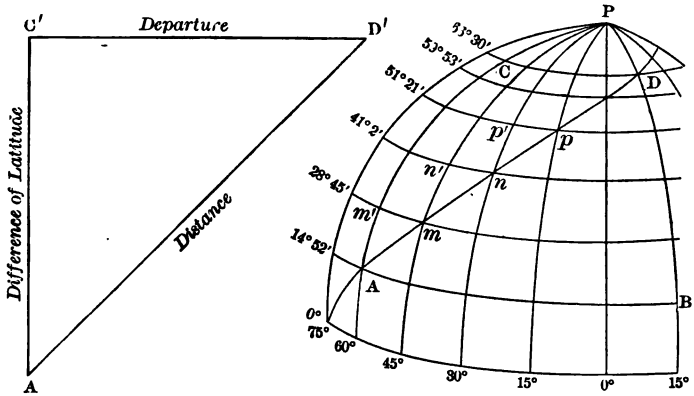
The *departure* is her easting or westing from her first meridian; it is measured as follows: if she sail on a parallel of latitude, the departure is the distance made on the parallel; if she sail on a loxodrome, the departure for each successive instant is measured on the parallel she is then crossing, and the limit of the sum of these infinitesimal departures is the total departure.

The unit of length is the *nautical mile*, about 6076 feet, a sixtieth part of a degree of a great circle of the earth. Sixty nautical miles are a little more than sixty-nine statute miles.

In what follows the earth is regarded as a perfect sphere. The error thus introduced is too small to be taken into account in any calculations whose data are derived from the log-line and compass.

PLANE SAILING. — RELATIONS BETWEEN COURSE, DISTANCE, DIFFERENCE OF LATITUDE, AND DEPARTURE.

Let AD be the rhumb-line, PA, PD the first and last meridians, Pm, Pn ⋯ meridians at equal small intervals;



let $m'm$, $n'n$, ⋯ be the small arcs intercepted on successive parallels;

then the total departure from A to D is the limit of the sum $m'm + n'n + \cdots$, when the meridians are taken very close together. [df. dep.

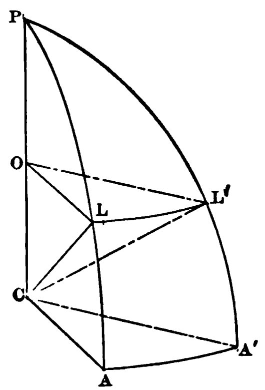
The infinitesimal triangles Amm', mnn' may be treated as right plane triangles; and since the course is constant they are similar. The elements of the motion are thus given by a series of infinitesimal right plane triangles, the sum of whose hypothenuses is the distance, of whose bases is the departure, and of whose altitudes is the difference of latitude. These three sums, and the course, have the same relations to each other as the parts of any one of the elemental triangles; hence they may be accurately represented by the parts of a right plane triangle. For this reason, although the sphericity of the earth is taken into account, the term *plane sailing* may be applied to any problem into which the difference of longitude does not enter; and the solution is effected by the rules for the solution of right plane triangles.

PARALLEL SAILING.—RELATIONS BETWEEN A DISTANCE SAILED ON A GIVEN PARALLEL OF LATITUDE AND THE DIFFERENCE OF LONGITUDE.

THEOR. 10. *The length of an arc of a parallel of latitude is the product of the length of the equatorial arc of the same number of degrees by the cosine of the latitude of the parallel.*

For let P be a pole of the earth, C its centre, A, A′ any two points on the equator, PA, PA′ two meridians cutting a parallel of latitude in L, L′;

let O be the centre of the arc LL′;



then ∵ arc LL′ : arc AA′ = OL : CA [geom

= OL : CL = cos ACL = cos. lat.,

∴ LL′ = AA′ cos. lat. Q. E. D.

MIDDLE LATITUDE SAILING.—APPROXIMATE RELATION BETWEEN THE DIFFERENCE OF LONGITUDE AND THE DEPARTURE ON A LOXODROME.

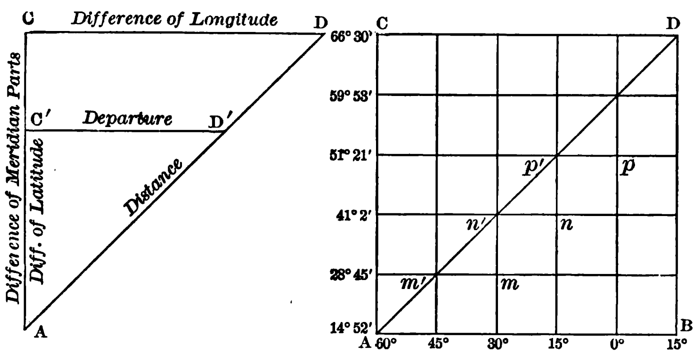
The departure from A to D lies, in value, between AB and CD, and for short distances is nearly the same as the ship makes if she sail between the same two meridians on the mid-parallel,

i.e., the parallel whose latitude is half the sum of the latitudes of A and D. Hence the departure from A to D is taken equal to the product of the difference of longitude of A and D by the cosine of their middle latitude. [th. 10

The difference of longitude is thus connected with the other elements of the ship's path.

MERCATOR'S PROJECTION. — ACCURATE RELATION BETWEEN THE DIFFERENCE OF LONGITUDE AND THE DEPARTURE ON A LOXODROME.

Project the figure of p. 129 on a plane surface as follows:



1. Draw a horizontal line for the equator and vertical lines at equal intervals for the meridians.

It follows that the projection $m'm$ of any arc of a parallel is equal to the corresponding arc of the equator, and is therefore multiplied by a projecting factor, the secant of its own latitude.

2. Draw a straight line cutting the meridians at the constant angle given by the course.

It follows that the angles of each small plane triangle remain the same; so that while each triangle is enlarged, its shape is preserved, and $m'n'$ or mn' has the same projecting factor as mm', and $n'p'$ or np' the same projecting factor as $n'n$, and so on.

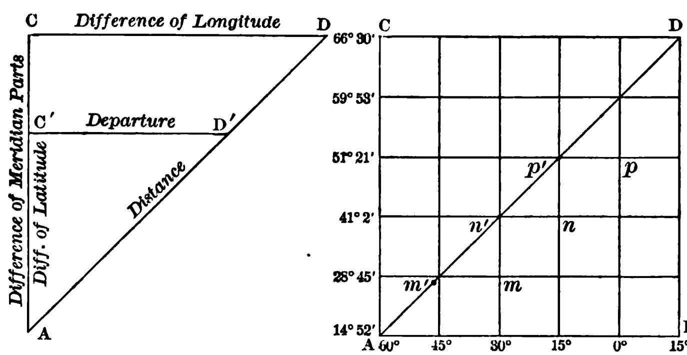
Each small portion of the meridian in the neighborhood of any parallel is therefore multiplied by the secant of the latitude of that parallel, and the total length of the projection of any given portion of a meridian is the limit of the sum of these products, when the parts are taken indefinitely small. In practice it is sufficiently accurate to take each part as two minutes or nautical miles, and to use as its projecting factor the secant of the latitude of its middle point.

E.g., the meridian-arc between the equator and latitude 13° 16′ projects into a distance on the chart equal to the sum

$$2(\sec 1' + \sec 3' + \sec 5' + \cdots + \sec 795')$$

in nautical miles, on the assumed scale. This distance is computed and tabulated as the *meridional part* for 13° 16′. In computing such a table, each entry may be used in succession to find the next one, *e.g.*, the meridional part for 36′ is found from that for 34′ by adding 2 sec 35′, and so on.

The difference between the meridional parts for two latitudes is their *meridional difference* of latitude.



In the figure above, AC is the meridional difference of latitude from A to D, and CD is the difference of longitude;

and dif. long. : merid. dif. lat. = tan course

= dep. : true dif. lat. [plane sailing

These equations connect the difference of longitude with the other elements of the ship's motion:

E.g., given the latitude and longitude of A, the course and distance from A to D;

find by plane sailing the departure, the difference of latitude, and the latitude of D;

find the meridional difference of latitude by subtracting meridional part for latitude A from that for latitude D;

compute the difference of longitude from the above relations.

NOTE. The student of the calculus will see that the exact meridional part for latitude λ is

$$\int_0^\lambda \sec\lambda \cdot d\lambda = \log_e \tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\lambda\right), \text{ in radians;}$$

and this result may be reduced to nautical miles, as follows:

$$\because \log_e \tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\lambda\right) = \log_{10} \tan\left(45^\circ + \tfrac{1}{2}\lambda\right) \cdot 2.3026,$$

and $1^r = 3437.75'$, $2.3026 \cdot 3437.75 = 7916$,

$$\therefore \log_e \tan\left(45^\circ + \tfrac{1}{2}\lambda\right) = \log_{10} \tan\left(45^\circ + \tfrac{1}{2}\lambda\right) \cdot 7916.$$

E.g., if $\lambda = 13^\circ\, 15'$;

then $45^\circ + \tfrac{1}{2}\lambda = 51^\circ\, 37'\, 30''$,

the merid. part $= 0.10134 \cdot 7916 = 802$ nautical miles,

and this latitude is enlarged in the ratio 802 : 795.

TRAVERSE SAILING.

PROB. 3. TO REDUCE THE RESULT OF SEVERAL SUCCESSIVE COURSES AND DISTANCES TO A SINGLE COURSE AND DISTANCE.

(*a*) *The latitude of the starting-point not given:*

Compute each separate difference of latitude and departure by plane sailing; take the algebraic sum of the separate differences of latitude for the value of the direct difference of latitude, and the algebraic sum of the departures for an approximate value of the direct departure; find the direct course and distance by plane sailing.

(*b*) *The latitude of the starting-point given:*

Compute the separate differences of $\left\{ \begin{array}{l} \textit{latitude} \\ \textit{longitude} \end{array} \right.$ *by Mercator's or middle-latitude sailing; take their algebraic sum for the direct difference of* $\left\{ \begin{array}{l} \textit{latitude;} \\ \textit{longitude;} \end{array} \right.$ *from these find the direct departure by Mercator's or middle-latitude sailing; find the direct course and distance by plane sailing.*

NOTE. The first course and distance entered is usually got by *taking a departure, i.e.*, by taking the bearing and distance of some object of known latitude and longitude; the reverse of these are entered on the log-slate as the first course and distance.

GREAT CIRCLE SAILING.

The shortest distance between two places is the great circle arc joining them; it does not cut all the meridians at the same angle; hence to keep on a great circle the ship must continually change her course. By means of a chart several places on the great circle may be determined, and if the ship lay her course for these on successive rhumb-lines, her path will differ little from the circular arc.

The *elements* of the great circle track between two given places are the distance, the first and last courses, and the highest latitude passed through. These are got from the spherical triangle whose vertical angle at the pole is the difference of longitude of the two places, and whose sides are their co-latitudes.

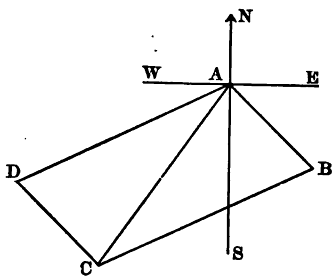
CURRENTS.

In order to ascertain the *set* and *drift* of a current, *i.e.*, its direction and velocity, a boat is taken a short distance from the ship and kept stationary by letting down a heavy weight; the log is thrown from the boat, and the direction in which it is carried, *i.e.*, the set of the current, is taken by the boat compass, while the drift is given by the number of knots run off in half a minute. The effect of the current is considered equivalent to an independent course.

E.g., if a ship sail 10 knots an hour in a current setting S.E. 5 miles an hour, what course must she lay to make a place whose bearing is S.W. by S.?

(*a*) *By construction and measurement.*

Take AB pointing S.E., and equal to 5 on any scale; take AC pointing S.W. by S.;



with B as centre and radius BC equal to 10 cut AC in the point C; complete the parallelogram ABCD:

the angle SAD is the course sought.

(*b*) *By computation.*

In the triangle ABC, the sides AB, BC and the angle BAC being known, compute the angle BCA, and thence the course SAD.

TACKING.

A ship is on the $\left\{\begin{matrix}\textit{starboard}\\\textit{port}\end{matrix}\right.$ *tack* when the wind is on her $\left\{\begin{matrix}\text{right;}\\\text{left;}\end{matrix}\right.$ she is *close-hauled* on either tack when she sails as nearly as possible toward the point whence the wind blows.

If when close-hauled she find her destination lying between her path and the wind, then she cannot reach it on this single tack; but she may continue till the angle that the direction of

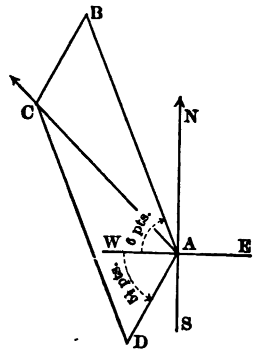
her destination makes with the wind is just equal to her angle of close-haul, and then run in close-hauled on the other tack.

E.g., if a ship can sail within 6 points of the wind on the port tack, and within $5\frac{1}{2}$ points on the starboard tack;

find her course and distance on each tack to reach, in the shortest time, a point 15 miles N.W., with the wind due west.

(*a*) *By construction and measurement.*

Take AC pointing N.W., and equal to 15 on any scale;
for the port tack draw AB 6 points to the right of the wind;
for the starboard tack draw AD $5\frac{1}{2}$ points to the left of the wind;
from the point C draw CB parallel to DA;
measure AB and BC for the distances on each tack.



(*b*) *By computation.*

In the triangle ABC, $AC = 15$, $A = 6 - 4 = 2$ pts,

$B = \overline{8-6} + \overline{8-5\frac{1}{2}} = 4\frac{1}{2}$ pts, $C = 5\frac{1}{2} + 4 = 9\frac{1}{2}$ pts,

[check $A + B + C = 16$ pts]; compute AB, BC.

The answer is the same whichever tack be taken first.

NOTE. The surface of the earth is supposed to be flat within the limits of these problems. They come usually under case (*b*) in compound-course sailing.

EXAMPLES.

1. A ship sails due west 117 miles from a point in lat. 38° N., long. 16° E.; find the longitude reached. [13° 31′ 30″ E.

2. In what latitude is a degree of longitude half as long as at the equator?

3. Sail S.E. 67 miles from New York light, lat. 40° 28′ N., long. 74° 8′ W.; by middle-latitude sailing find the latitude and longitude of the point reached.
[39° 40′ 36″ N., 73° 6′ 6″ W.

4. Find the course and distance from Montauk Point, 41° 4′ N., 72° W., to Martha's Vineyard, 41° 17′ N., 70° 48′ W.
[N. 76° 30′ 40″ E., 55.73 miles by middle-latitude sailing; N. 76° 43′ E., 56.58 miles by Mercator's sailing.

5. A ship sails from a point 14° 45′ N., 17° 33′ W., on a course S. 28° 7′ 30″ W., till she reaches longitude 29° 26′ W.; find by Mercator the distance sailed and the latitude.
[1500 miles, 7° 18′ S.

6. From a point in latitude 50° 10′ S. a ship sails S. 67° 30′ E. till her departure is 957 miles; find by Mercator the distance sailed, the difference of latitude, and the difference of longitude. [1036, 6° 36′ 24″, 26° 53′.

7. A ship starting from a point in latitude 32° N. sails N. 25° E. 16 miles, thence S. 54 E. 11 miles, thence N. 13 W. 7 miles, thence N. 61° E. 5 miles, thence N. 38° W. 18 miles; find the single course and distance that would bring her to the same destination. [(*a*) N. 13° 12′ 20″ E., 32.30 miles;
(*b*) N. 11° 40′ 37″ E., 32.12 miles.

8. Find the elements of the great circle track between New York light and Cape Clear, 51° 26′ N., 9° 29′ W.

9. So, between San Francisco, 37° 48′ N., 122° 25′ W., and Cape of Good Hope, 33° 56′ S., 18° 29′ E.

*V. DERIVATIVES, SERIES, TABLES, Etc.

§ 1. DERIVATIVES OF TRIGONOMETRIC FUNCTIONS.

For definition of *limit*, *derivative*, etc., see O. W. J. alg. VII.

Some of the fundamental properties of derivatives are, for convenience of reference, set down here as lemmas without proof; they are given in two forms:

If A, B be functions of any same variable x, then:

LEM. 1. $D_x(A+B)=D_xA+D_xB$, $\delta(A+B)\doteq\delta A+\delta B$.

LEM. 2. $D_x(A\cdot B)=B\cdot D_xA+A\cdot D_xB$, $\delta(A\cdot B)\doteq B\cdot\delta A+A\cdot\delta B$.

LEM. 3. $D_x(A:B)=(B\cdot D_xA-A\cdot D_xB):B^2$,

$\delta(A:B)\doteq(B\cdot\delta A-A\cdot\delta B):B_2$.

LEM. 4. $D_xA^n=n\cdot A^{n-1}\cdot D_xA$, $\delta A^n\doteq n\cdot A^{n-1}\cdot\delta A$.

LEM. 5. $D_x\log_e A=D_xA:A$, $\delta\log_e A\doteq\delta A:A$.

wherein $D_x\equiv$ *x-derivative of*, $\delta\equiv$ *an increment of*, and the sign $\doteq$, read *approaches*, means that the difference of the two members is infinitesimal as to either of them.

THEOR. 1. *If θ be the measure of an infinitesimal plane angle, in radians, then* $lim\ (sin\,\theta:\theta)=1$, $lim\ (tan\,\theta:\theta)=1$.

For, let c be the circumference of a circle, O the centre;

let p, p' be the perimeters of two regular polygons of the same number of sides, the first inscribed in and the other circumscribed about the circle, and having their pairs of sides PP′, TT′, ⋯ parallel;

draw OAX perpendicular to PP′, TT′;

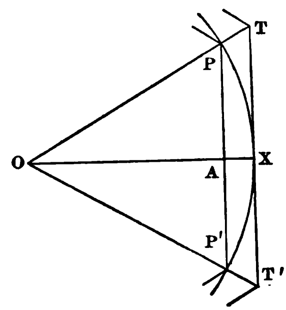
then $\because$ $p<c<p'$, whatever the number of sides, [geom.

and $p\doteq p'$ when the number of sides becomes infinite,

$\therefore$ c is the common limit of p, p',

$\therefore$ unity is the common limit of the ratios $p:c$, $p':c$;

and $\because$ the two polygons are similar, and AP, XP, XT are like parts of p, c, p', [geom.



$\therefore$ $\sin\theta : \theta, = \frac{\text{AP}}{\text{OP}} : \frac{\text{XP}}{\text{OP}}, = \text{AP} : \text{XP} = p : c,$

and $\because$ $\tan\theta : \theta, = \frac{\text{XT}}{\text{OX}} : \frac{\text{XP}}{\text{OX}}, = \text{XT} : \text{XP} = p' : c;$

$\therefore$ $\lim(\sin\theta : \theta) = \lim(p : c) = 1,$

and $\lim(\tan\theta : \theta) = \lim(p' : c) = 1.$ Q. E. D.

COR. *lim (chord* $\theta : \theta) = \lim(2\sin\tfrac{1}{2}\theta : \theta) = 1.$

THEOR. 2. *If θ be any plane angle, then:*

$$\text{D}_\theta \sin\theta = \cos\theta, \qquad \text{D}_\theta \csc\theta = -\cot\theta\csc\theta,$$

$$\text{D}_\theta \cos\theta = -\sin\theta, \qquad \text{D}_\theta \sec\theta = \tan\theta\sec\theta,$$

$$\text{D}_\theta \tan\theta = \sec^2\theta, \qquad \text{D}_\theta \cot\theta = -\csc^2\theta.$$

For, let θ' be an infinitesimal angle, the increment of θ;

then $\because$ $\sin(\theta+\theta') - \sin\theta = 2\cos(\theta+\tfrac{1}{2}\theta')\sin\tfrac{1}{2}\theta'$, [II th. 12

$\therefore$ $[\sin(\theta+\theta') - \sin\theta] : \theta' = \cos(\theta+\tfrac{1}{2}\theta')\cdot\sin\tfrac{1}{2}\theta' : \tfrac{1}{2}\theta'.$

But $\because$ θ' is the increment of θ, [hyp.

and $\sin(\theta+\theta') - \sin\theta$ is the consequent increment of $\sin\theta$,

$\therefore$ $\lim(\text{inc}\sin\theta : \text{inc}\,\theta), \equiv \text{D}_\theta\sin\theta, = \cos\theta.$ Q. E. D. [th. 1

So $\because$ $\cos(\theta+\theta')-\cos\theta = -2\sin(\theta+\frac{1}{2}\theta')\sin\frac{1}{2}\theta'$,

$\therefore$ $[\cos(\theta+\theta')-\cos\theta]:\theta' = -\sin(\theta+\frac{1}{2}\theta')\cdot\sin\frac{1}{2}\theta':\frac{1}{2}\theta'$,

$\therefore$ $\lim(\text{inc}\cos\theta:\text{inc}\,\theta)$, $\equiv \text{D}_\theta\cos\theta$, $= -\sin\theta$. Q.E.D. [th. 1

So $\text{D}_\theta\tan\theta$, $\equiv \text{D}_\theta(\sin\theta:\cos\theta)$,

$$= (\cos\theta\,\text{D}_\theta\sin\theta - \sin\theta\,\text{D}_\theta\cos\theta):\cos^2\theta$$

$$= (\cos^2\theta+\sin^2\theta):\cos^2\theta = \sec^2\theta.$$

So for $\text{D}_\theta\csc\theta$, $\equiv \text{D}_\theta(1:\sin\theta)$, for $\text{D}_\theta\sec\theta$, for $\text{D}_\theta\cot\theta$.

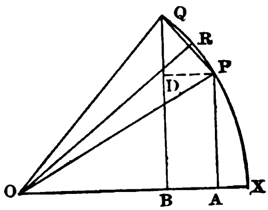
Geometric proof. Let O-XP be any circle, and Q a point on this circle near P;

bisect arc PQ at R, and join OX, OP, OQ, OR;

draw ordinates AP, BQ;

join P, Q, and through P draw a parallel to OX meeting BQ in D;

let $\theta\equiv\angle$ XOP, $\theta'\equiv\angle$ POQ, $(\theta+\frac{1}{2}\theta')\equiv\angle$ XOR, $r\equiv$ radius of circle;



then $\because$ $\sin\theta = \text{AP}:r$, $\sin(\theta+\theta') = \text{BQ}:r$, $\theta' = \text{arc PQ}:r$,

and $\angle$ DQP $= \angle$ XOR, [geom.

$\therefore$ $[\sin(\theta+\theta')-\sin\theta]:\theta' = \text{DQ}:\text{arc PQ}$

$$= (\text{DQ}:\text{ch. PQ})\cdot(\text{ch. PQ}:\text{arc PQ})$$

$$= \cos(\theta+\tfrac{1}{2}\theta')\cdot(\text{ch. PQ}:\text{arc PQ}),$$

$\therefore$ $\lim(\text{inc}\sin\theta:\text{inc}\,\theta)$, $\equiv \text{D}_\theta\sin\theta$, $= \cos\theta$; [th. 1 cr.

Q. E. D.

and so for $\text{D}_\theta\cos\theta$, $\text{D}_\theta\tan\theta$, $\cdots$.

EXAMPLES.

1. If θ be any plane angle and θ' be the increment of θ:

then
$$\operatorname{inc}^2 \sin\theta = -(2\sin\tfrac{1}{2}\theta')^2 \sin(\theta+\theta'),$$
$$\operatorname{inc}^2 \cos\theta = -(2\sin\tfrac{1}{2}\theta')^2 \cos(\theta+\theta'),$$
$$\operatorname{inc}^4 \sin\theta = \quad (2\sin\tfrac{1}{2}\theta')^4 \sin(\theta+2\theta'),$$
$$\operatorname{inc}^4 \cos\theta = \quad (2\sin\tfrac{1}{2}\theta')^4 \cos(\theta+2\theta'),$$

wherein $\operatorname{inc}^2 \sin\theta \equiv$ the increment of the increment of $\sin\theta$,

i.e., $[\sin(\theta+2\theta') - \sin(\theta+\theta')] - [\sin(\theta+\theta') - \sin\theta]$,

or $\sin(\theta+2\theta') - 2\sin(\theta+\theta') + \sin\theta$;

and $\operatorname{inc}^4 \sin\theta \equiv \operatorname{inc}\operatorname{inc}\operatorname{inc}\operatorname{inc}\sin\theta$,

i.e.,
$$\sin(\theta+4\theta') - 4\sin(\theta+3\theta') + 6\sin(\theta+2\theta') - 4\sin(\theta+\theta') + \sin\theta.$$

2. If δp, δq, δr, $\delta \mathrm{P}$, $\delta \mathrm{Q}$, be any simultaneous small changes in the values of p, q, r, P, Q, that are consistent with the known relations of the parts of a right triangle

$$[\mathrm{P}+\mathrm{Q}=90^\circ,\ p^2+q^2=r^2,\ p=r\sin\mathrm{P},\ q=r\cos\mathrm{P}],$$

then
$$\delta\mathrm{Q} = -\delta\mathrm{P},\ \delta r \doteq \frac{p}{r}\cdot\delta p + \frac{q}{r}\cdot\delta q = \sin\mathrm{P}\cdot\delta p + \cos\mathrm{P}\cdot\delta q,$$
$$\delta p \doteq \sin\mathrm{P}\cdot\delta r + r\cos\mathrm{P}\cdot\delta\mathrm{P},\ \delta q \doteq \cos\mathrm{P}\cdot\delta r - r\sin\mathrm{P}\cdot\delta\mathrm{P},$$

and [eliminate δr from the last two equations]

$$\delta\mathrm{P} \doteq \frac{\cos\mathrm{P}}{r}\cdot\delta p - \frac{\sin\mathrm{P}}{r}\cdot\delta q = \frac{q\delta p - p\delta q}{p^2+q^2}.$$

3. If, in a right triangle, only the values of p, q be given, and if these have the possible errors ${}^{\pm}p'$, ${}^{\pm}q'$; *i.e.*, if p may possibly differ from its assumed value by either ${}^{+}p'$ or ${}^{-}p'$, and q by either ${}^{+}q'$ or ${}^{-}q'$; show from ex. 2 that the resulting values of r, P will have the possible errors

$$\pm\frac{pp'+qq'}{r},\ = \pm(p'\sin\mathrm{P} + q'\cos\mathrm{P}),$$

and
$$\pm\frac{pq'+qp'}{r^2},\ = \pm\frac{1}{r}(q'\sin\mathrm{P} + p'\cos\mathrm{P});$$

wherein p', q' are positive.

So, if only q, r be given, with the possible errors $^{\pm}q'$, $^{\pm}r'$, find the possible errors of the other sides and angles.

So, if only q, P be given, or only r, P, with the possible errors $^{\pm}q'$, $^{\pm}\text{P}'$, or $^{\pm}r'$, $^{\pm}\text{P}'$.

4. From the known relations of the parts of an oblique triangle [$\text{A}+\text{B}+\text{C}=180°$, $a\sin\text{B}=b\sin\text{A}, \cdots$] prove that

(a) $\delta\text{A}+\delta\text{B}+\delta\text{C}=0,$

(b) $b\cos\text{A}\cdot\delta\text{A}-a\cos\text{B}\cdot\delta\text{B}-\sin\text{B}\cdot\delta a+\sin\text{A}\cdot\delta b=0,$

$c\cos\text{B}\cdot\delta\text{B}-b\cos\text{C}\cdot\delta\text{C}-\sin\text{C}\cdot\delta b+\sin\text{B}\cdot\delta c=0,$

$a\cos\text{C}\cdot\delta\text{C}-c\cos\text{A}\cdot\delta\text{A}-\sin\text{A}\cdot\delta c+\sin\text{C}\cdot\delta a=0.$

From these equations, by elimination and reduction, derive

(c) $b\cdot\delta\text{C}+c\cos\text{A}\cdot\delta\text{B}-\sin\text{A}\cdot\delta c+\sin\text{C}\cdot\delta a=0,$

$c\cdot\delta\text{B}+b\cos\text{A}\cdot\delta\text{C}-\sin\text{A}\cdot\delta b+\sin\text{B}\cdot\delta a=0,$

with four symmetric equations;

(d) $b\sin\text{C}\cdot\delta\text{A}-\delta a+\cos\text{C}\cdot\delta b+\cos\text{B}\cdot\delta c=0,$

with two symmetric equations.

5. If in an oblique triangle only a, B, C be given, and if their possible errors be $^{\pm}a : 10000$, $^{\pm}10''$, $^{\pm}15''$, find the possible errors of A [ex. 4, a]; of b [ex. 4, c]; of c [ex. 4, c].

Find the values of these possible errors when ABC is very nearly equilateral, 5000 feet on each side.

6. Given the values of C, a, b, with the possible errors $^{\pm}\text{C}'$, $^{\pm}a'$, $^{\pm}b'$, find the possible errors of B, A, c [ex. 4, c, d].

7. Given A, a, b, with possible errors $^{\pm}\text{A}'$, $^{\pm}a'$, $^{\pm}b'$, find the possible errors of B [ex. 4, b]; of C, c.

8. Given A, B, b, with possible errors $^{\pm}\text{A}'$, $^{\pm}\text{B}'$, $^{\pm}b'$, find the possible errors of C, a, c; first, when, as in all the above cases, the computation is assumed to be exact; second, when C, a, c have the further possible errors $^{\pm}\text{C}''$, $^{\pm}a''$, $^{\pm}c''$ from decimal figures omitted in the computation.

9. Given a, b, c, with possible errors $^{\pm}a'$, $^{\pm}b'$, $^{\pm}c'$, find the possible error of A; with a possible error in computation of $^{\pm}\text{A}''$.

§ 2. EXPANSION OF TRIGONOMETRIC FUNCTIONS.

THEOR. 3. *If θ be any plane angle expressed in radians, then:*

$$\sin\theta = \theta - \frac{\theta^3}{3\,!} + \frac{\theta^5}{5\,!} - \frac{\theta^7}{7\,!} + \cdots,$$

$$\cos\theta = 1 - \frac{\theta^2}{2\,!} + \frac{\theta^4}{4\,!} - \frac{\theta^6}{6\,!} + \cdots.$$

For, let $\sin\theta = \text{A} + \text{B}\theta + \text{C}\theta^2 + \text{D}\theta^3 + \text{E}\theta^4 + \text{F}\theta^5 + \cdots,$

wherein A, B, C, ⋯ are unknown, but constant;

find the successive derivatives of both members of the equation;

then $\cos\theta = \text{B} + 2\text{C}\theta + 3\text{D}\theta^2 + 4\text{E}\theta^3 + 5\text{F}\theta^4 + \cdots;$ [th. 2

$$-\sin\theta = 2\text{C} + 6\text{D}\theta + 12\text{E}\theta^2 + 20\text{F}\theta^3 + \cdots;$$

$$-\cos\theta = 6\text{D} + 24\text{E}\theta + 60\text{F}\theta^2 + \cdots;$$

$$\sin\theta = 24\text{E} + 120\text{F}\theta + \cdots;$$

$$\cos\theta = 120\text{F} + \cdots.$$

If θ be made 0,

then $\text{A} = 0,\ \text{B} = 1,\ \text{C} = 0,\ \text{D} = -1 : 3\,!,\ \text{E} = 0,\ \text{F} = 1 : 5\,!,\ \cdots.$

Replace A, B, C, ⋯ by their values,

then $\sin\theta = \theta - \dfrac{\theta^3}{3\,!} + \dfrac{\theta^5}{5\,!} - \dfrac{\theta^7}{7\,!} + \cdots.$ Q. E. D.

Find the derivative of each member of this equation,

then $\cos\theta = 1 - \dfrac{\theta^2}{2\,!} + \dfrac{\theta^4}{4\,!} - \dfrac{\theta^6}{6\,!} + \cdots.$ Q. E. D.

NOTE 1. That $\sin\theta$ can be expanded into a series of the form $\text{A} + \text{B}\theta + \text{C}\theta^2 + \text{D}\theta^3 + \cdots$, is assumed in the above proof.

[O. W. J. alg. XII th. 27, nt.

NOTE 2. These series are convergent for all finite values of θ.

For the ratios of successive terms are:

$$\frac{\theta^2}{2\cdot3},\ \frac{\theta^2}{4\cdot5},\ \frac{\theta^2}{6\cdot7},\ \cdots;\quad \frac{\theta^2}{1\cdot2},\ \frac{\theta^2}{3\cdot4},\ \frac{\theta^2}{5\cdot6},\ \cdots,$$

i.e., series of fractions that become smaller than some constant that is smaller than unity, whatever the value of θ,

and the radius of convergence is ∞. [O. W. J. alg. XII th. 11

COR. 1. $$tan\,\theta = \theta + \frac{\theta^3}{3} + \frac{2\,\theta^5}{3\cdot 5} + \frac{17\,\theta^7}{3^2\cdot 5\cdot 7} + \frac{62\,\theta^9}{3^4\cdot 5\cdot 7} + \cdots,$$

$$cot\,\theta = \frac{1}{\theta} - \frac{\theta}{3} - \frac{\theta^3}{3^2\cdot 5} - \frac{2\,\theta^5}{3^3\cdot 5\cdot 7} - \frac{\theta^7}{3^3\cdot 5^2\cdot 7} - \cdots,$$

$$sec\,\theta = 1 + \frac{\theta^2}{2} + \frac{5\,\theta^4}{2^3\cdot 3} + \frac{61\,\theta^6}{2^4\cdot 3^2\cdot 5} + \frac{277\,\theta^8}{2^7\cdot 3^2\cdot 7} + \cdots,$$

$$csc\,\theta = \frac{1}{\theta} + \frac{\theta}{2\cdot 3} + \frac{7\,\theta^3}{2^3\cdot 3^2\cdot 5} + \frac{31\,\theta^5}{2^4\cdot 3^3\cdot 5\cdot 7} + \frac{127\,\theta^7}{2^7\cdot 3^3\cdot 5^2\cdot 7} +$$

For the tangent, divide the value of sine by that of cosine;
for the cotangent, divide the value of cosine by that of sine;
for the secant, divide unity by the value of cosine;
for the cosecant, divide unity by the value of sine.

NOTE. The series for $\tan\theta$, $\sec\theta$ are convergent only when $\theta < \frac{1}{2}\pi$; for they are finite and continuous for all values of θ smaller than $\frac{1}{2}\pi$, but when $\theta = \frac{1}{2}\pi$ their values are infinite and the radius of convergence is $\frac{1}{2}\pi$; so the series for $\theta\cot\theta$, $\theta\csc\theta$ are convergent only when $\theta < \pi$.

COR. 2. $$log\text{-}sin\,\theta = log\,\theta - \frac{\theta^2}{2\cdot 3} - \frac{\theta^4}{2^2\cdot 3^2\cdot 5} - \frac{\theta^6}{3^4\cdot 5\cdot 7} - \cdots$$

$$log\text{-}cos\,\theta = -\left(\frac{\theta^2}{2} + \frac{\theta^4}{2^2\cdot 3} + \frac{\theta^6}{3^2\cdot 5} + \frac{17\,\theta^8}{2^3\cdot 3^2\cdot 5\cdot 7} + \cdots\right).$$

For $\because\ D_\theta \log\text{-}\sin\theta = \dfrac{\cos\theta}{\sin\theta} = \cot\theta = \dfrac{1}{\theta} - \dfrac{\theta}{3} - \dfrac{\theta^3}{3^2\cdot 5} - \cdots,$ [lm.

$\therefore\ \log\text{-}\sin\theta = \log\theta - \dfrac{\theta^2}{2\cdot 3} - \dfrac{\theta^4}{2^2\cdot 3^2\cdot 5} - \dfrac{\theta^6}{3^4\cdot 5\cdot 7} - \cdots,$ [lm.

i.e., log-sin $\theta =$ the series whose θ-derivative is the above series for cot θ, and which, as $\theta \doteq 0$, approaches to log θ as log-sin θ must do.

So $\because\ D_\theta \log\text{-}\cos\theta = -\dfrac{\sin\theta}{\cos\theta} = -\tan\theta = -\left(\theta + \dfrac{\theta^3}{3} + \cdots\right),$

$\therefore\ \log\text{-}\cos\theta = -\left(\dfrac{\theta^2}{2} + \dfrac{\theta^4}{2^2\cdot 3} + \dfrac{\theta^6}{3^2\cdot 5} + \dfrac{17\,\theta^8}{2^3\cdot 3^2\cdot 5\cdot 7} + \cdots\right).$

NOTE. The series for log-sin θ is convergent for all values of θ smaller than π; that for log-cos θ for all values smaller than $\frac{1}{2}\pi$.

§ 3. COMPUTATION OF FUNCTIONS.

Prob. 1. To compute a table of natural sines and cosines.

(*a*) *For angles* $0^\circ \cdots 30^\circ$: *replace* θ *by* $1', 2', 3', \cdots$ *in the formulæ of th.* 3.

E.g., $\because\ 1' = \dfrac{\pi}{180 \times 60} = \dfrac{3.141\,592\,653\,589\,793}{10\,800}$

$$= .000\,290\,888\,208\,666,$$

$$\therefore\ \sin 1' = .000\,290\,888\,208\,666 - \frac{.000\,290\,888\,208\,666^3}{3\,!} + \cdots$$

$$= .000\,290\,8882;$$

$$\cos 1' = 1 - \frac{.000\,290\,888\,208\,666^2}{2\,!} + \cdots$$

$$= .999\,999\,9577;$$

$$\sin 2' = 2 \times .000\,290\,888\,208\,666 - 2^3 \times \frac{.000\,290\,888\,208\,666^3}{3\,!} + \cdots$$

$$= .000\,581\,7764;$$

$$\cos 2' = 1 - 2^2 \times \frac{.000\,290\,888\,208\,666^2}{2\,!} + \cdots$$

$$= .999\,999\,8308.$$

Note 1. The fraction $\pi : 10{,}800$ once raised to the required powers, first, second, third, $\cdots$, and divided by the factorials $1\,!, 2\,!, 3\,!, \cdots$, thereafter only simple multiples of the quotients are used. At first but two terms of the series are needed; but later, when θ is larger, and the series therefore converges less rapidly, more terms must be taken.

E.g., for 30°, $\theta = \frac{1}{6}\pi = .52360$ nearly;

and
$$\sin 30^\circ = .52360 - \frac{.5236^3}{6} + \frac{.5236^5}{120} - \frac{.5236^7}{5040} + \cdots$$

$$= .52360 - .02392 + .00033 - .00000 + \cdots$$

$$= .5 \text{ within less than } .00005;$$

i.e., by the use of three terms of the series, the sine is found correct to four decimal places, the same degree of accuracy as that assumed for the value of π.

(*b*) *For angles* $30° \cdots 45°$: *replace* θ' *by* $1', 2', 3', \cdots$ *in formulæ :*

$sin(30°+\theta') = cos\,\theta' - sin(30°-\theta')$, [ad. th., $\sin 30° = \frac{1}{2}$

$cos(30°+\theta') = cos(30°-\theta') - sin\,\theta'$.

E.g.,
$$\begin{aligned}\sin 30°\,1' &= \cos 1' - \sin 29°\,59'\\ &= .999\,999\cdots - .499\,75 = .500\,25,\\ \sin 30°\,2' &= \cos 2' - \sin 29°\,58'\\ &= 999\,999\cdots - .499\,50 = .500\,50,\\ \cos 30°\,1' &= \cos 29°\,59' - \sin 1'\\ &= .866\,17 - .000\,29 = .865\,88,\\ \cos 30°\,2' &= \cos 29°\,58' - \sin 2' = .865\,73.\end{aligned}$$

(*c*) *For angles* $45° \cdots 90°$: *apply formulæ :*

$sin\,(45° + \theta') = cos\,(45° - \theta')$; [II th. 8

$cos\,(45° + \theta') = sin\,(45° - \theta')$.

E.g.,
$$\begin{aligned}\sin 45°\,1' &= \cos 44°\,59' = .707\,31,\\ \sin 45°\,2' &= \cos 44°\,58' = .707\,52,\\ \cos 45°\,1' &= \sin 44°\,59' = .706\,90,\\ \cos 45°\,2' &= \sin 44°\,58' = .706\,70.\end{aligned}$$

Note 2. Verification. The results are tested in many ways :

(*a*) $\because\ \sin \frac{1}{2}\theta = \sqrt{\dfrac{1-\cos\theta}{2}}, \quad \cos\frac{1}{2}\theta = \sqrt{\dfrac{1+\cos\theta}{2}},$ [II th. 14

$\therefore$ from $\cos 45°, = \sqrt{\frac{1}{2}}$, are found in succession the sines and cosines of $22°\,30'$, $11°\,15'$, $\cdots$.

So from $\cos 30°, = \frac{1}{2}\sqrt{3}$, are found in succession the sines and cosines of $15°$, $7°\,30'$, $\cdots$.

(*b*) $\because\ \sin 2\theta = 2\sin\theta\cos\theta,$ [II th. 14

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta,$ [II th. 16 cr.

and $\sin 36° = \cos 54°,$ [II th. 8

$\therefore\ 2\sin 18°\cos 18° = 4\cos^3 18° - 3\cos 18°,$

$\therefore\ 2\sin 18° = 4\,(1 - \sin^2 18°) - 3,$

$\therefore\ \sin 18° = \frac{1}{4}(\sqrt{5} - 1), \quad \cos 18° = \frac{1}{4}\sqrt{(10 + 2\sqrt{5})}$;

thence in succession the sines and cosines of $9°$, $4°\,30'$, $2°\,15'$, $\cdots$.

(*c*) From $\cos 36° = \cos^2 18° - \sin^2 18° = \frac{1}{4}(\sqrt{5}+1)$,

and $\sin 36° = \sqrt{(1-\cos^2 36°)} = \frac{1}{4}\sqrt{(10-2\sqrt{5})}$,

are found the sine and cosine of $(36° - 30°)$, *i.e.*, of 6°, thence in succession the sine and cosine of 3°, 1° 30′, 45′, ⋯.

(*d*) From $\sin(36° + \theta') - \sin(36° - \theta'), = 2\cos 36° \sin\theta'$
$$= \tfrac{1}{2}(\sqrt{5}+1)\sin\theta',$$

subtract $\sin(72° + \theta') - \sin(72° - \theta'), = 2\cos 72° \sin\theta'$
$$= \tfrac{1}{2}(\sqrt{5}-1)\sin\theta',$$

then,
$$\sin(36° + \theta') - \sin(36° - \theta')$$
$$= \sin(72° + \theta') - \sin(72° - \theta') + \sin\theta',$$

a formula (Euler's) that serves to test the sines of all angles from 0° to 90°, if to θ' be given the different values from 0° to 18°.

In the same way may be used either of the test formulæ found in II § 8, exs. 9–12.

PROB. 2. TO COMPUTE TABLES OF NATURAL TANGENTS, COTANGENTS, SECANTS, AND COSECANTS.

Divide the sines of the angles, in order, by the cosines, each by each; the cosines by the sines; unity by the cosines; unity by the sines:

or, replace θ *by* 1′, 2′, 3′, ⋯ *in the formulæ of th.* 3, *cr.* 1.

PROB. 3. TO COMPUTE TABLES OF LOGARITHMIC FUNCTIONS.

From a table of logarithms of numbers take out the logarithms of the natural sines and cosines:

or, replace θ *by* 1′, 2′, 3′, ⋯ *in the formulæ of th.* 3, *cr.* 2.

Subtract the logarithmic cosines from the logarithmic sines; the logarithmic sines from the logarithmic cosines; the logarithmic cosines and sines from 0.

NOTE. 1. A more rapid method, applicable also to making tables of natural functions, and many others, is this:

Take out the functions of three, four, or more angles at regular intervals, and find their several "orders of differences"; then, by the algebraic "method of differences," find the successive terms of

the series of logarithms, and interpolate for other angles lying between those of the series. Verify at intervals by direct computation.
[O. W. J. alg. XII §§ 10, 11

For safety, four-place tables must be computed to six places; five-place tables to seven places, and so on.

When the terms of any order of differences are constant, or differ very little, the rule that follows may be applied to form new terms of the series:

Add the constant difference to the last difference of the next lower order, that sum to the last difference of the next lower order, and so on till a term of the series is reached.

In the example that follows, the numbers below the heavy rules are got by successive addition:

angle.	log sine	1st dif.	2d dif.	3d dif.
18°	9.4899824			
		38689		
18° 10′	9.4938513		−378	
		38311		7
18° 20′	9.4976824		−371	
		37940		7
18° 30′	9.5014764		−364	
		37576		7
18° 40′	9.5052340		−357	
		37219		7
18° 50′	9.5089559		−350	
		36869		
19°	9.5126428			

EXAMPLES.

1. Compute the natural sines and cosines of:

$22° 30'$, $67° 30'$, $11° 15'$, $78° 45'$, ⋯,

$15°$, $75°$, $7° 30'$, $82° 30'$, ⋯,

$6°$, $84°$, $3°$, $87°$, ⋯,

$9°$, $81°$, $4° 30'$, $85° 30'$, ⋯.

2. By interpolation in the table above find the logarithmic sines of $18° 1'$, $18° 2'$, $18° 3'$, $18° 4'$, ⋯.

3. From the logarithmic sines of 8°, 8° 10′, 8° 20′, 8° 30′, taken from the table, find the several orders of differences, thence find three more terms of the series.

4. Interpolate to minutes between 8° 10′ and 8° 20′.

§ 4. RELATIONS BETWEEN PLANE AND SPHERICAL TRIANGLES.

After the definition of the trigonometric functions and the statement of their relations, all the properties of the right spherical triangle, and of the plane triangle (oblique and right), may be derived from those of the oblique spherical triangle. Such a development of the subject presents the principles of trigonometry in their most general form, and teaches the student to take these general propositions and, by successive steps, to draw out and state in their logical order the special propositions that are included in them. This mutual relation of the general and the particular not only helps the intellect to grasp these propositions, but also helps the memory to retain them.

The order of development is this:

To state and prove the general properties of the oblique spherical triangle, counting it the most general form of the triangle.

To derive the properties of the right spherical triangle, counting it a special case of the oblique spherical triangle, wherein one angle is a right angle.

To derive the general properties of the oblique plane triangle, counting it a special case of the spherical triangle, wherein the radius of the sphere has become infinite and the arcs straight lines.

To derive the properties of the right plane triangle, counting it a special case of the oblique plane triangle, wherein one of the angles is a right angle; or of the right spherical triangle wherein the arcs are straight lines.

The "law of sines" and the "law of cosines" are proved directly, as follows:

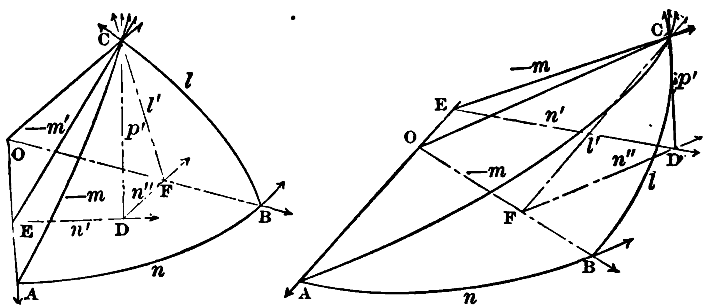
Let l, m, n be any three directed arcs on a sphere, centre at O, forming a spherical triangle ABC.

let the reader so place himself before this triangle that the positive end of the arc n is to the right of A, and positive rotation about A is counter-clock-wise.

through C draw p' perpendicular to the plane OAD and directed up from D in that plane;

through p' draw a plane perpendicular to OA at E and cutting the planes of the arcs n, ${}^-m$ in n', ${}^-m'$, normals to OA;

and a plane perpendicular to OB at F, and cutting the planes of the arcs n, l in n'', l', normals to OB;



then will p' be normal to n', n'',

and $\qquad \angle n'\,{}^-m' = \angle n\,{}^-m = \angle \text{A}, \quad \angle n''l' = \angle nl = \sup \angle \text{B},$

$\qquad \angle \text{BOA} = \text{arc}\ {}^-c, \qquad \angle \text{OB}n' = \frac{1}{2}\pi - c.$

(*a*) $\sin \text{A} : \sin \text{B} = \sin a : \sin b.$

For $\because$ $\text{DC} = \text{EC} \sin \text{A} = \text{OC} \sin b \sin \text{A},$

and $\text{DC} = \text{FC} \sin \sup \text{B} = \text{OC} \sin a \sin \text{B},$

$\therefore$ $\sin b \cdot \sin \text{A} = \sin a \cdot \sin \text{B},$

$\therefore$ $\sin \text{A} : \sin \text{B} = \sin a : \sin b.$ Q.E.D.

(b) $\cos a = \cos b \cos c + \sin b \sin c \cos \mathrm{A}$.

For $\because \mathrm{OF} = \mathrm{OC} \cos a$,

and $\mathrm{OF} = \mathrm{OE} \cos \overline{-c} + \mathrm{ED} \cos \overline{\tfrac{1}{2}\pi - c}$

$= \mathrm{OC} \cos b \cos c + \mathrm{OC} \sin b \cos \mathrm{A} \sin c$,

$\therefore \cos a = \cos b \cos c + \sin b \sin c \cos \mathrm{A}$. Q. E. D.

(c) $\cos \mathrm{A} = - \cos \mathrm{B} \cos \mathrm{C} + \sin \mathrm{B} \sin \mathrm{C} \cos a$.

For $\because \sin^2 a \sin b \sin c (\cos \mathrm{A} + \cos \mathrm{B} \cos \mathrm{C})$

$= \sin^2 a (\cos a - \cos b \cos c)$

$+ (\cos b - \cos c \cos a)(\cos c - \cos a \cos b)$ [(b)

$= \cos a (1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c)$,

and $\sin^2 a \sin b \sin c \cdot \sin \mathrm{B} \sin \mathrm{C} \cos a$

$= \sin^2 b \sin^2 c \sin^2 \mathrm{A} \cos a$ [(a)

$= \cos a [\sin^2 b \sin^2 c - (\cos a - \cos b \cos c)^2]$, [(b)

$= \cos a (1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c)$,

$\therefore \cos \mathrm{A} + \cos \mathrm{B} \cos \mathrm{C} = \sin \mathrm{B} \sin \mathrm{C} \cos a$. Q. E. D.

The derivation of other formulæ is as given in IV § 5, and that of the special properties of the right spherical triangle from these is outlined in IV § 5 ex. 2.

The general properties of the plane triangle may be got from those of the spherical triangles as follows:

If the sides of a spherical triangle subtend very small angles at the centre of the sphere, the spherical triangle differs but little from a plane triangle having the same vertices; and, if the vertices be fixed in position while the centre of the sphere recedes further and further away, and the radii grow longer, then the bounding arcs grow straighter, the spherical triangle approaches closer to the plane triangle having the same vertices, the small angles at the centre of the sphere, subtended by the sides of the triangle, are proportional to those sides, and the sum of the three angles of the triangle is a little greater than, but approaches, two right angles.

The plane triangle that has the same vertices is the limit to which the spherical triangle approaches when the radius is infinite, and if in the formulæ for spherical triangles the functions of the sides be expressed in terms of their subtended angles, and only those terms be retained whose limiting ratios are finite, *i.e.*, those that are of the same lowest order of infinitesimal, the resulting formulæ correspond to the formulæ for plane triangles.

In detail: replace $\sin a$, $\cos a$, $\tan a$, $\cdots$ by $a - \frac{a^3}{3!} + \cdots$,

$$1 - \frac{a^2}{2!} + \cdots, \quad a + \frac{a^3}{3} + \cdots, \text{ and so on};$$

omit all terms except those of lowest order, and replace these infinitesimals by the corresponding sides of the plane triangle.

(*a*) *The terms of lowest degree of the first order:*

Replace $\sin a$, $\tan a$, $2 \sin \frac{1}{2} a$, $\cdots$ by a, $\cos a$ by 1, and so on;

then $\because \sin p = \sin r \sin \text{P} = \tan q \cot \text{Q}$,

$\therefore p = r \sin \text{P} = q \cot \text{Q}$;

and $\because \cos \text{P} = \cos p \sin \text{Q} = \tan q \cot r$,

$\therefore \cos \text{P} = 1 \cdot \sin \text{Q} = q : r$;

and $\because \sin a : \sin b = \sin \text{A} : \sin \text{B}$,

$\therefore a : b = \sin \text{A} : \sin \text{B}$;

and $\because \sin \frac{1}{2}\text{A} = \sqrt{[\sin (s-b) \sin (s-c) : \sin b \sin c]}$,

$\therefore \sin \frac{1}{2}\text{A} = \sqrt{[(s-b)(s-c) : bc]}$;

and $\because \cos \frac{1}{2}\text{A} = \sqrt{[\sin s \sin (s-a) : \sin b \sin c]}$,

$\therefore \cos \frac{1}{2}\text{A} = \sqrt{[s(s-a) : bc]}$;

and $\because \tan \frac{1}{2}\text{A} = \sqrt{[\sin (s-b) \sin (s-c) : \sin s \sin (s-a)]}$,

$\therefore \tan \frac{1}{2}\text{A} = \sqrt{[(s-b)(s-c) : s(s-a)}$;

and $\because \sin \frac{1}{2}(\text{A}-\text{B}) = \sin \frac{1}{2}(a-b) : \sin \frac{1}{2}c \cdot \cos \frac{1}{2}\text{C}$,

$\therefore \sin \frac{1}{2}(\text{A}-\text{B}) = (a-b) : c \cdot \cos \frac{1}{2}\text{C}$;

and $\because \cos \frac{1}{2}(\text{A}-\text{B}) = \sin \frac{1}{2}(a+b) : \sin \frac{1}{2}c \cdot \sin \frac{1}{2}\text{C}$,

$\therefore \cos \frac{1}{2}(\text{A}-\text{B}) = (a+b) : c \cdot \sin \frac{1}{2}\text{C}$;

and $\because \tan\frac{1}{2}(\text{A}-\text{B}) = \sin\frac{1}{2}(a-b) : \sin\frac{1}{2}(a+b) \cdot \cot\frac{1}{2}\text{C}$,

$\therefore \tan\frac{1}{2}(\text{A}-\text{B}) = (a-b) : (a+b) \cdot \cot\frac{1}{2}\text{C}$;

and $\because \tan\frac{1}{2}(a+b) = \cos\frac{1}{2}(\text{A}-\text{B}) : \cos\frac{1}{2}(\text{A}+\text{B}) \cdot \tan\frac{1}{2}c$,

$\therefore a+b = \cos\frac{1}{2}(\text{A}-\text{B}) : \cos\frac{1}{2}(\text{A}+\text{B}) \cdot c$;

and $\because \tan\frac{1}{2}(a-b) = \sin\frac{1}{2}(\text{A}-\text{B}) : \sin\frac{1}{2}(\text{A}+\text{B}) \cdot \tan\frac{1}{2}c$,

$\therefore a-b = \sin\frac{1}{2}(\text{A}-\text{B}) : \sin\frac{1}{2}(\text{A}+\text{B}) \cdot c$.

(*b*) *The terms of lowest degree of the second order:*

Replace $\sin a$, $\tan a$, by a; $\cos a$ by $1-\frac{1}{2}a^2$; and so on.

E.g., the formula $\cos r = \cos p \cos q$ becomes

$$1-\tfrac{1}{2}r^2+\cdots = (1-\tfrac{1}{2}p^2+\cdots)(1-\tfrac{1}{2}q^2+\cdots)$$
$$= 1-\tfrac{1}{2}p^2-\tfrac{1}{2}q^2+\cdots;$$

$\therefore p^2+q^2 = r^2 \pm$ terms of higher degree, whose ratios to p^2, q^2, $r^2 \doteq 0$ when p, q, $r \doteq 0$,

$\therefore p^2+q^2=r^2$.

So, the formula $\cos a = \cos b \cos c + \sin b \sin c \cos \text{A}$ gives

$$a^2 = b^2+c^2-2\,bc\cos\text{A}.$$

EXAMPLES.

1. Show that of exs. 1–7, IV § 3, exs. 1, 2 reduce, for a plane right triangle, to $q^2+p^2=r^2$;
   exs. 3, 4, to $\tan\frac{1}{2}\text{P} = \sqrt{[(r-q):(r+q)]} = p:(r+q)$;
   ex. 5, to $p-q = p\tan\frac{1}{2}\text{P} - q\tan\frac{1}{2}\text{Q}$;
   exs. 6, 7, to $\text{P}+\text{Q}=90°$.

2. Show that of exs. 3–14, IV § 5, ex. 3 reduces, for a plane triangle, to ex. 1, III § 4;
   ex. 7, to ex. 2, III § 4;
   ex. 5, to ex. 4, III § 4;
   ex. 9, to $(s-c):(s-a) = \tan\frac{1}{2}\text{A} : \tan\frac{1}{2}\text{C}$.

   Show what the other examples of IV § 5 reduce to.

LEGENDRE'S THEOREM. — GEODESY.

THEOR. 4. *If* ABC *be any spherical triangle whose sides are very small as to the radius of the sphere, and if* A′B′C′ *be a plane triangle whose sides* a', b', c' *are equal in absolute length to the sides* a, b, c *of the spherical triangle, then each angle* A, B, C *exceeds the corresponding angle* A′, B′, C′ *by one-third of the spherical excess of the triangle* ABC.

Replace $\sin b$, $\sin c$ by $b - \frac{1}{6}b^3 + \cdots$, $c - \frac{1}{6}c^3 + \cdots$,
and $\cos a$, $\cos b$, $\cos c$ by $1 - \frac{1}{2}a^2 + \cdots$, $1 - \frac{1}{2}b^2 + \cdots$, $1 - \frac{1}{2}c^2 + \cdots$;
then $\because$ the formula $\cos a = \cos b \cos c + \sin b \sin c \cos A$ gives

$$bc\,(\cos A' - \cos A) \qquad [\cos A' \equiv (b^2 + c^2 - a^2) : 2bc$$
$$= \tfrac{1}{12}(a^2b^2 + b^2c^2 + c^2a^2) - \tfrac{1}{24}(a^4 + b^4 + c^4) \pm \text{terms}$$

whose ratios to these terms $\doteq 0$, when $a, b, c \doteq 0$,

and so for $ca\,(\cos B' - \cos B)$, $ab\,(\cos C' - \cos C)$, [sym.

$$\therefore\ bc(\cos A' - \cos A) \doteq ca(\cos B' - \cos B) \doteq ab(\cos C' - \cos C).$$

But $\because\ \cos A' - \cos A = 2 \sin\frac{1}{2}(A - A')\sin\frac{1}{2}(A + A')$

$$\doteq (A - A')\sin A',$$

and so for $\cos B' - \cos B$, $\cos C' - \cos C$, [sym.

$$\therefore\ bc\,(A - A')\sin A' \doteq ca\,(B - B')\sin B' \doteq ab\,(C - C')\sin C,$$
$$\therefore\ A - A' \doteq B - B' \doteq C - C' \doteq \tfrac{1}{3}[(A + B + C) - (A' + B' + C')].$$

Q. E. D.

EXAMPLES.

3. Triangles upon the earth's surface are regarded as spherical triangles, and the earth's mean radius is 3956 miles. If the angles A, B be 65°, 60° and the side c be 100 miles, find the sides a, b in degrees and in miles; find the angle C, and the spherical excess; find the area of the triangle in square miles; find the number of square miles that correspond to 1″ of spherical excess.

4. In a geodetic survey $\angle A = 30°$, $\angle B = 48° 45'$, $\angle C = 101° 15' 12''$, side $c = 70$ miles; find the angles of the plane triangle whose sides equal a, b, c of the spherical triangle, and thence find the lengths of a, b. [Leg. th.

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